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Letter

Real-commutant phase selection from real Hilbert representations

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ABSTRACT

Complex Hilbert space is usually assumed at the start of quantum mechanics. This paper formulates a reverse construction for the one-particle and operator-algebraic setting. Starting from a real Hilbert space carrying an orthogonal representation, the admissible imaginary units are identified with the skew-adjoint square roots of $-\mathrm{Id}$ in the Real commutant of the complexified representation. The novelty is not the isolated use of complexification, spectral orientation, or polar decomposition, but their order of dependence: first the symmetry representation defines an intrinsic operator-valued phase space; then an oriented internal time flow selects a distinguished point of that phase space; only afterward are the usual complex one-particle, observable, probability, and Weyl–CCR structures invoked under their standard hypotheses. In the terminology used here, this oriented internal flow is the reversible or inner-generative side of a dual-time distinction. It is not a second observer clock and not an additional Lorentzian coordinate: mathematically it is the strongly continuous orthogonal group $U(t) = \exp(tA)$ on the real Hilbert space. Its Stone-dual variable is energy/frequency λ; canonical conjugation pairs λ with $-\lambda$; positive energy orients this pair and selects the canonical phase on the dynamic sector through

$$A = J|A| = |A|J.$$

Thus inner-generative time is the source of phase orientation, while irreversible record time, projection, memory, and decoherence enter only as later reduced-dynamics layers. No spectral gap at zero is required, and zero modes remain visible as residual phase data rather than being hidden by the construction. The selected phase is invariant under time-orientation-preserving symmetries and changes sign under time reversal, giving the usual antiunitary action after complex reconstruction. The paper also makes explicit the monoidal limitation: two selected local phases do not make the raw real tensor product the standard composite system; the standard composite is recovered only after imposing the J-balanced, equivalently complex, tensor product rule. Thus the result is a reconstruction of compatible complex kinematics on the selected one-particle/operator-algebraic sector, not a derivation of all quantum foundations from purely real data.

1. Introduction

The complex scalar field is one of the first postulates in the usual Hilbert-space formulation of quantum mechanics. State vectors are complex, observables are self-adjoint complex-linear operators, and transition amplitudes are complex numbers. A different order is possible. One may start from a real Hilbert space with orthogonal symmetry and ask whether the imaginary unit can be identified as an operator compatible with that symmetry.

An orthogonal complex structure on a real Hilbert space $H_{\mathbb{R}}$ is a bounded real-linear operator J satisfying

$$J^* = -J, \qquad J^2 = -\mathrm{Id}. \tag{1}$$

Such an operator turns $H_{\mathbb{R}}$ into a complex Hilbert space by declaring multiplication by i to be J. The problem is therefore not whether one can write a complex scalar formally, but which operators J are intrinsically available and which, if any, is selected by the given physical structure.

The role of time in this question is central. The paper uses the phrase *dual-time* in a restricted mathematical sense: temporal structure is separated into a reversible, inner-generative role and an irreversible, record-forming role. The reversible role is the one used in the theorems. It is the internal one-parameter group $t \mapsto U(t)$ that generates motion on the real Hilbert space. Its generator A is real and skew-adjoint; its Stone-dual spectral parameter λ is energy or frequency. Canonical conjugation pairs λ and $-\lambda$. A positive orientation of the inner-generative flow therefore chooses one side of this paired spectrum, and the main theorem proves that this orientation is exactly the operator that becomes multiplication by i. The irreversible or record-forming role–semigroups, memory kernels, coarse-graining, decoherence, or measurement records–is not used

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to select J; it can only act after the J-linear event algebra has been fixed [1,2].

This question has a long background in real-Hilbert formulations of quantum theory, beginning with Stueckelberg's work [3]. The positive-frequency construction itself is also familiar from oscillator and Klein–Gordon quantization, and Segal's Hilbert-space and CCR viewpoint is part of the standard background [4]. Moretti and Oppio proved a particularly strong relativistic result: for elementary systems on real Hilbert space, Poincare symmetry together with positivity can force a natural invariant complex structure, unique up to sign [5]. The present paper does not claim that the sign of a time generator, polar decomposition, or positive-frequency splitting is new in isolation. Its contribution is the prior classification and dependency step: before any time orientation is used, all symmetry-compatible imaginary units are organized as square roots of $-\mathrm{Id}$ in a Real commutant. Positive energy is then a selection principle acting on that already-defined phase space.

A closely related recent real-Hilbert-space formulation is given by Singh [6], who rewrites Schrödinger dynamics on a real space and represents multiplication by i by a fixed symplectic complex structure $\mathcal{J}$, with $\mathcal{J}^2 = -\mathrm{Id}$. The relation to the present J_A is most transparent at the level of the real generator. In that realification the Schrödinger equation has the form $\partial_t \mathbf{U} = A\mathbf{U}$ with $A = \mathcal{J}H/\hbar$, where H is the realified self-adjoint Hamiltonian and $\mathcal{J}H = H\mathcal{J}$. If $H \geq 0$ on the sector under consideration, then $|A| = H/\hbar$, and uniqueness of polar decomposition gives $J_A = \mathcal{J}$ on the dynamic sector. More generally, $J_A = \mathcal{J}\,\mathrm{sgn}(H)$ on the nonzero spectral sector, so the two structures coincide after the positive-energy orientation has been fixed. Singh's $\mathcal{J}$ is therefore the complex structure built into a chosen real representation of complex Schrödinger dynamics, whereas the present construction first classifies all symmetry-compatible complex structures in the Real commutant and only then selects the polar phase of an oriented generator. The comparison therefore concerns this classification-before-selection order, not a different positive-frequency complex structure.

The phase-selection architecture is summarized by the following reconstruction chain:

$$\text{real orthogonal symmetry} \longrightarrow \text{Real-commutant phase space},$$
$$\text{inner-generative time flow } U(t) = e^{tA} \longrightarrow \text{dual energy orientation } \lambda > 0,$$
$$\text{selected phase } J_A \longrightarrow \text{complex quantum kinematics}.$$

The individual ingredients–commutants, complexification, direct integrals, spectral orientations, and polar decomposition–are standard. Their ordering and packaging define the phase-selection architecture used here. Complex phase is first made an operator-valued Real-commutant invariant; the inner-generative time flow then orients the Stone-dual energy pair; and only afterward are the usual complex observable, probability, and Weyl–CCR structures reconstructed under the usual hypotheses. In this precise sense, the mathematical content of the dual-time reading is not an extra coordinate but the dependency order

$$A \quad \rightsquigarrow \quad \mathrm{sgn}(K) \quad \rightsquigarrow \quad J_A \quad \rightsquigarrow \quad i.$$

The phase-selection construction should therefore be understood primarily as a mathematical reconstruction and selection theorem, not as an additional microscopic mechanism that produces complex numbers. Its physical input is the choice of a distinguished oriented time flow, equivalently a positive-energy orientation; given that input, the theorem identifies the unique polar phase on the dynamic sector. In the oscillator and stationary-field examples this selected operator is exactly the standard positive-frequency complex structure. This identifies the shared phase-selection core used in the focused developments below.

A broader mathematics-first framework containing the Real-commutant parameterization, positive-frequency polar selection, covariance, compact Frobenius–Schur analysis, and phase-balanced tensor construction was recently developed by the author [7]. That work embeds the shared phase-selection core in a larger architecture involving completion quotients, temporal spectral triples, and conditional cone geometry. The present Letter isolates the one-particle and operator-algebraic quantum-kinematic sector; conversely, those completion-quotient, spectral-triple, and cone-geometric modules are not used in the proofs below. The present work should therefore be read as a focused continuation and quantum-kinematic development of the shared phase-selection core, rather than as an independent first presentation of that core.

The novelty can be summarized in five points. First, the paper treats the imaginary unit as an operator-valued invariant of a real symmetry representation, not as a primitive scalar-field convention. Second, the admissible phases are classified before a positive-frequency orientation is chosen. Third, the type-I, abelian, and compact formulas are presented as coordinate models of the same Real-commutant invariant rather than as unrelated constructions. Fourth, the positive-energy theorem is interpreted as an inner-generative time theorem: the real generator A does not merely evolve already complex states; through its oriented dual spectrum it selects the complex phase itself. Fifth, the zero-mode and composite-system limitations are not suppressed: zero modes remain residual phase data, and standard composites require the J-balanced complex tensor product rather than the raw real tensor product.

This ordering distinguishes the present construction from neighboring approaches. Modular-localization approaches start from standard real subspaces and Tomita–Takesaki modular data to build local nets of observables from one-particle representations [8]. Geometric and Kähler formulations, such as Ashtekar–Schilling, start from an already complex Hilbert space or its projective ray space and then express quantum mechanics in symplectic-geometric terms [9]. The relation to such work is therefore indirect rather than a continuation of that program: here the question is earlier in the dependency chain, namely which real-linear operator supplies multiplication by i before the complex projective geometry or local net has been fixed.

The recent debate on real-number quantum mechanics also shows why the tensor product rule must be kept explicit. Renou et al. argued that real and complex Hilbert-space formulations can differ in network scenarios when the usual real tensor-product postulate is imposed [10]. More recent works propose real formulations that reproduce standard multipartite predictions by modifying or quotienting the composition rule rather than using the naive real tensor product [11,12]. The present paper is orthogonal to that debate in scope. It does not defend the raw real tensor product as a complete many-body composition law. It reconstructs the compatible complex phase on a selected one-particle/operator-algebraic sector, and Section 4.1 states explicitly how the standard complex tensor product is recovered only after a J-balanced monoidal rule is imposed.

This phase-selection problem is distinct from recent equivalence programs based on composition rules. Those works ask which composition postulate allows a real-valued theory to reproduce all multipartite predictions of complex quantum mechanics [11,12]. Here the question is earlier and single-system/operator-algebraic: which symmetry-compatible operators can play the role of i, and what oriented flow selects one? The Real commutant answers the first question and J_A answers the second; no operational equivalence of real and complex multipartite theories follows until an additional composition rule is supplied.

The result is conditional. It assumes a real Hilbert representation and, for the selection theorem, a distinguished strongly continuous time flow. It does not claim that every real representation admits a global complex structure, nor that the Born rule is derived without the usual probability assumptions. Once the complex structure has been selected, the observable algebra is the ordinary noncommutative complex Hilbert-space algebra, so the usual Bell and Kochen–Specker constraints remain [13,14]. Measurement is likewise not used to choose the phase. Records, decoherence, and effective collapse belong to a later projection or reduction layer after the selected J-linear event algebra has already been obtained.

The paper is organized as follows. Section 2 defines the Real commutant and proves the intrinsic classification of admissible phases, including the type-I field picture and the compact Frobenius–Schur blocks. Section 3 proves positive-energy selection by polar factorization and isolates the zero-mode phase problem. Section 4 explains covariance, complex reconstruction, composite-system compatibility, probability, the Weyl–CCR extension, and the later status of records. Section 5 gives examples with explicit Hilbert spaces and operator domains. Section 6 summarizes the scope and limitations, including the limitations for interacting quantum field theory.

2. The real commutant and admissible phases

Let G be a topological group and let $U : G \to O(H_{\mathbb{R}})$ be a strongly continuous orthogonal representation on a real Hilbert space. Let

$$H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$$

be the complexification, written as $x + iy$ with $x, y \in H_{\mathbb{R}}$. Let C denote canonical conjugation,

$$C(x + iy) = x - iy,$$

and let $V(g)$ be the complexified representation,

$$V(g)(x + iy) = U(g)x + iU(g)y.$$

Definition 1. [Real commutant] The complex commutant of U is

$$\mathcal{A}(U) = \{T \in B(H_{\mathbb{C}}) : TV(g) = V(g)T \text{ for all } g \in G\}. \tag{2}$$

The Real commutant is the real C^*-subalgebra

$$\mathcal{A}_{\mathbb{R}}(U) = \{T \in \mathcal{A}(U) : TC = CT\}. \tag{3}$$

The real commutant is

$$\mathrm{Comm}_{\mathbb{R}}(U) = \{S \in B(H_{\mathbb{R}}) : SU(g) = U(g)S \text{ for all } g \in G\}. \tag{4}$$

Definition 2 (Admissible phases)**.** The admissible symmetry-compatible complex structures are

$$\mathcal{J}(U) = \{J \in \mathrm{Comm}_{\mathbb{R}}(U) : J^* = -J,\ J^2 = -\mathrm{Id}\}. \tag{5}$$

Equivalently, after complexification, $\mathcal{J}(U)$ is regarded as a subset of $\mathcal{A}_{\mathbb{R}}(U)$.

Theorem 1 (Intrinsic commutant formulation)**.** *Complexification induces an isometric real $*$-isomorphism*

$$\mathrm{Comm}_{\mathbb{R}}(U) \longrightarrow \mathcal{A}_{\mathbb{R}}(U), \qquad S \longmapsto S_{\mathbb{C}}. \tag{6}$$

Consequently $\mathcal{J}(U)$ is canonically identified with the set of skew-adjoint square roots of $-\mathrm{Id}$ in $\mathcal{A}_{\mathbb{R}}(U)$:

$$\mathcal{J}(U) \cong \{J \in \mathcal{A}_{\mathbb{R}}(U) : J^* = -J,\ J^2 = -\mathrm{Id}\}. \tag{7}$$

Proof. If $S \in \mathrm{Comm}_{\mathbb{R}}(U)$, its complexification $S_{\mathbb{C}}$ is complex-linear, commutes with every $V(g)$, and commutes with C. Hence $S_{\mathbb{C}} \in \mathcal{A}_{\mathbb{R}}(U)$. This map preserves products, adjoints, and the operator norm. Conversely, if $T \in \mathcal{A}_{\mathbb{R}}(U)$, then $TC = CT$, so T preserves the fixed-point space of C, namely $H_{\mathbb{R}}$. Let $S = T|_{H_{\mathbb{R}}}$. Then S is bounded, real-linear, and commutes with $U(g)$ because T commutes with $V(g)$. Complex-linearity and preservation of $H_{\mathbb{R}}$ imply $T = S_{\mathbb{C}}$. The equations $J^* = -J$ and $J^2 = -\mathrm{Id}$ are transported by this isomorphism. ☐

Remark 1 (Intrinsic versus Type-I classification)**.** The definition of $\mathcal{A}_{\mathbb{R}}(U)$ and the phase space (7) do not require a type-I representation. For an arbitrary representation, $\mathcal{A}_{\mathbb{R}}(U)$ is a real von Neumann algebra and the admissible phases remain

$$\{J \in \mathcal{A}_{\mathbb{R}}(U) : J^* = -J,\ J^2 = -\mathrm{Id}\}.$$

The type-I hypothesis below is used only to obtain a computable field description. For type-II or type-III commutants the same question becomes the classification of orthogonal complex structures inside a general real W^*-algebra; this is a genuine extension problem, not an assumption of the intrinsic theorem.

The theorem gives a compact answer to the classification problem: the possible imaginary units compatible with the real symmetry are the square roots of $-\mathrm{Id}$ inside an intrinsic real operator algebra. The set may be empty, discrete, or infinite-dimensional. The following standard representation-theoretic forms make this explicit.

First, suppose that the complexified representation has a type-I direct-integral decomposition

$$H_{\mathbb{C}} \simeq \int_X^{\oplus} K_x \otimes M_x \, d\mu(x), \qquad V(g) \simeq \int_X^{\oplus} \pi_x(g) \otimes \mathrm{Id}_{M_x} \, d\mu(x), \tag{8}$$

with essentially disjoint irreducible fields. If canonical conjugation sends the fiber over x to the fiber over an involutive partner ιx, and if $c_x : M_x \to M_{\iota x}$ denotes the induced antiunitary transport on multiplicity spaces, then a decomposable commutant element

$$T \simeq \int_X^{\oplus} \mathrm{Id}_{K_x} \otimes A_x \, d\mu(x)$$

belongs to $\mathcal{A}_{\mathbb{R}}(U)$ exactly when

$$A_{\iota x} c_x = c_x A_x \tag{9}$$

for almost every x. Admissible phases are precisely the essentially bounded measurable fields satisfying (9) and

$$A_x^* = -A_x, \qquad A_x^2 = -\mathrm{Id}_{M_x} \tag{10}$$

for almost every x. The antiunitaries c_x are a measurable transport gauge: replacing them by a gauge-equivalent field conjugates the matrices A_x and gives an isomorphic Real commutant. Thus the intrinsic object is not the chosen field model but the gauge class of the Real multiplicity transport. This is the direct-integral form of Theorem 1, using the usual decomposable-commutant theorem for type-I representations [15–17].

For locally compact abelian G, the same structure becomes a paired spectral orientation. Let $\hat{G}$ be the Pontryagin dual and let

$$V(g) = \int_{\hat{G}} \chi(g) \, dE(\chi) \tag{11}$$

be the spectral representation [18]. Conjugation pairs χ with χ^{-1}. A Borel function $\sigma : \hat{G} \to \{-1, 0, 1\}$ satisfying $\sigma(\chi^{-1}) = -\sigma(\chi)$ defines

$$J_\sigma = i\big(E(\sigma^{-1}(1)) - E(\sigma^{-1}(-1))\big)\Big|_{H_{\mathbb{R}}}. \tag{12}$$

On the sector for which $\sigma \neq 0$, J_σ is an orthogonal complex structure commuting with U. For $G = \mathbb{R}$, the choice $\sigma(\lambda) = \mathrm{sgn}(\lambda)$ is the usual positive-frequency split.

For compact G, Peter–Weyl theory and the Frobenius–Schur trichotomy give a finite or countable block description [19–21]. If P is a set of representatives of complex-type pairs, $\hat{G}_{\mathrm{re}}$ the real-type self-conjugate irreducibles, and $\hat{G}_{\mathrm{qu}}$ the quaternionic-type self-conjugate irreducibles, then

$$\mathcal{A}_{\mathbb{R}}(U) \simeq \prod_{\pi \in P} B_{\mathbb{C}}(M_\pi) \times \prod_{\rho \in \hat{G}_{\mathrm{re}}} B_{\mathbb{R}}(M_\rho^{\mathrm{re}}) \times \prod_{\tau \in \hat{G}_{\mathrm{qu}}} B_{\mathbb{H}}(M_\tau^{\mathrm{qu}}). \tag{13}$$

Admissible phases are blockwise skew-adjoint square roots of $-\mathrm{Id}$. In finite dimension this immediately shows the only elementary obstruction: every real-type multiplicity block must have even real dimension. Complex-type and quaternionic-type multiplicity blocks carry such square roots without this even-real-dimension obstruction.

In this sense the classification is constructive in the standard decomposition regimes. In the abelian multiplicity-one case one computes an odd measurable sign on the paired spectrum. In a finite-dimensional compact or finite-group problem one decomposes the complexified representation into Frobenius–Schur blocks and then solves finite matrix equations in the corresponding factors of (13). Real-type blocks of odd real multiplicity are exactly the elementary obstruction; complex-type and quaternionic-type blocks admit blockwise square roots of $-\mathrm{Id}$. In the type-I direct-integral case the same procedure becomes a measurable field problem subject to the transport relation (9). Thus the theorem is descriptive in complete generality but computational in the usual spectral, compact, and type-I settings.

3. Positive-energy selection

Classification does not yet select a physical phase. A real representation may carry many compatible complex structures. A distinguished time evolution supplies the additional datum. This is where the inner-generative time role enters the paper mathematically.

Definition 3 (Dual-time and inner-generative time used here)**.** For the purposes of this paper, a dual-time reading is a separation of two mathematical roles, not the postulate of two observable clocks.

1. The *inner-generative* or reversible role is the parameter t in the strongly continuous orthogonal group $U(t)$. Its real skew-adjoint generator is A, and its Stone-dual spectral parameter is λ, interpreted as energy or frequency.
2. Canonical conjugation exchanges the dual energies λ and $-\lambda$. A time orientation, equivalently the choice $\lambda > 0$, orients this paired spectrum.
3. The irreversible, record, or effective role would be represented by additional data such as a contraction semigroup, a Lindblad generator, a memory kernel, a projected age variable, or a record quotient. This second role is not an input to the proof of phase selection.

Thus positive energy selects the reversible orientation of inner-generative time. No additional macroscopic clock, extra Lorentzian direction, or independent time coordinate is introduced.

Let $U : \mathbb{R} \to O(H_{\mathbb{R}})$ be a strongly continuous orthogonal flow with skew-adjoint generator A,

$$U(t) = \exp(tA), \qquad A^* = -A. \tag{14}$$

Set

$$H_0 = \ker A, \qquad H_{\mathrm{dyn}} = (\ker A)^{\perp} = \overline{\operatorname{Ran} A}. \tag{15}$$

The fixed sector H_0 is not discarded as a mistake; it is simply the sector on which the time orientation carries no sign. No spectral gap at zero is assumed.

On $H_{\mathbb{C}}$, Stone's theorem gives a self-adjoint operator K such that [22,23]

$$A_{\mathbb{C}} = iK, \qquad U_{\mathbb{C}}(t) = \exp(itK). \tag{16}$$

Since $A_{\mathbb{C}}$ commutes with the canonical conjugation C, anti-linearity gives $CKC = -K$. Hence the spectral measure of K is paired by $\lambda \mapsto -\lambda$.

Theorem 2 (Positive-energy Polar Phase)**.** *On H_{dyn} there is a unique orthogonal complex structure J_A satisfying*

$$A = J_A B = B J_A, \qquad B = |A| = (A^* A)^{1/2} \geq 0, \tag{17}$$

with $\operatorname{Dom}(B) = \operatorname{Dom}(A) \cap H_{\mathrm{dyn}}$*, as equalities of unbounded operators on* $\operatorname{Dom}(A) \cap H_{\mathrm{dyn}}$*. Equivalently, if E is the spectral measure of K, then*

$$J_A = i\big(E((0,\infty)) - E((-\infty,0))\big)\Big|_{H_{\mathbb{R}}} \tag{18}$$

with $J_A = 0$ on H_0. On H_{dyn}, $J_A^* = -J_A$ and $J_A^2 = -\mathrm{Id}$.

Proof. The closed skew-adjoint operator A has polar decomposition $A = W|A|$, where the partial isometry W has initial space $\overline{\operatorname{Ran}|A|} = (\ker A)^{\perp}$ and final space $\overline{\operatorname{Ran} A} = (\ker A)^{\perp}$. Thus W is orthogonal on H_{dyn}. On the complexification, $A_{\mathbb{C}} = iK$ and $|A_{\mathbb{C}}| = |K|$. Functional calculus gives the polar phase $i\,\operatorname{sgn}(K)$ on $E(\mathbb{R} \setminus \{0\})H_{\mathbb{C}}$, whose restriction to $H_{\mathbb{R}}$ is exactly (18). The identities $J_A^* = -J_A$, $J_A^2 = -\mathrm{Id}$ on H_{dyn}, and $J_A B = B J_A$ follow from the same spectral calculus. Uniqueness follows from uniqueness of the polar decomposition on the support of $|A|$. □

The theorem recovers the positive-frequency complex structure without assuming complex scalars at the outset. Positive spectrum of K is assigned multiplication by i, negative spectrum multiplication by $-i$, and the real fixed vectors are obtained by pairing the two spectral halves through canonical conjugation. The selection is intrinsic to the oriented time flow.

In the terminology of Definition 3, A is the inner-generative time generator, $|A|$ is the positive rate or energy operator, and J_A is the orientation operator of the paired dual spectrum. The complex phase is therefore not an independent scalar postulate at this layer; it is the polar orientation of real inner-generative dynamics.

Corollary 1 (Uniqueness in the Positive Factorization Class)**.** *Let $\widetilde{J}$ be an orthogonal complex structure on H_{dyn} and let $\widetilde{B} \geq 0$ be self-adjoint with* $\operatorname{Dom}(\widetilde{B}) = \operatorname{Dom}(A) \cap H_{\mathrm{dyn}}$ *such that, as equalities of unbounded operators on this domain,*

$$A = \widetilde{J}\widetilde{B} = \widetilde{B}\widetilde{J}. \tag{19}$$

Then $\widetilde{B} = |A|$ and $\widetilde{J} = J_A$.

Proof. Eq. (19) is a polar decomposition of A on the support of its absolute value, with the same operator domain as A restricted to H_{dyn}. By uniqueness of polar decomposition, the positive factor and the polar phase agree with those in Theorem 2. □

Remark 2 (Zero Modes as Residual Phase Data)**.** If $\ker A \neq 0$, time orientation does not select a complex structure on H_0. This is a residual phase problem, not a technical failure. If a residual symmetry representation U_0 acts on H_0, the remaining compatible choices are exactly

$$\mathcal{J}(U_0) = \{J_0 \in \mathrm{Comm}_{\mathbb{R}}(U_0) : J_0^* = -J_0,\ J_0^2 = -\mathrm{Id}\}.$$

Whenever the splitting $H_{\mathbb{R}} = H_0 \oplus H_{\mathrm{dyn}}$ is reducing for the relevant symmetry, every $J_0 \in \mathcal{J}(U_0)$ produces a full phase $J_0 \oplus J_A$. In constrained systems the zero sector may instead be quotiented or treated as gauge. In systems with genuine static degrees of freedom–for example the zero mode of a massless scalar field on a compact spatial manifold–additional physical data such as a symplectic form, boundary condition, infrared prescription, or state must be supplied. Thus the polar construction is canonical precisely on H_{dyn}, while H_0 records the missing input.

4. Covariance and quantum reconstruction

The phase selected by a time subgroup need not commute with every larger symmetry. It commutes exactly with the symmetries that preserve the chosen time orientation.

Theorem 3 (Covariance of the selected phase)**.** *Let W be a real unitary on $H_{\mathbb{R}}$ such that $W\operatorname{Dom}(A) = \operatorname{Dom}(A)$ and*

$$WAW^{-1} = aA \tag{20}$$

for some nonzero real number a. Then $WH_{\mathrm{dyn}} = H_{\mathrm{dyn}}$ and, on H_{dyn},

$$WJ_AW^{-1} = \operatorname{sgn}(a)J_A, \qquad W|A|W^{-1} = |a|\,|A|. \tag{21}$$

Consequently time-orientation-preserving symmetries are complex-linear for the reconstructed complex structure, while time-orientation-reversing symmetries are antiunitary after reconstruction.

Proof. Eq. (20) implies $W\ker A = \ker A$, hence $WH_{\mathrm{dyn}} = H_{\mathrm{dyn}}$. It also implies $|WAW^{-1}| = |a|\,|A|$. Conjugating $A = J_A|A|$ gives

$$aA = (WJ_AW^{-1})(|a|\,|A|). \tag{22}$$

Uniqueness of polar decomposition on H_{dyn} yields (21). If $a > 0$, then W commutes with J_A and is complex-linear. If $a < 0$, then $WJ_A = -J_AW$, which is exactly anti-linearity relative to multiplication by $i = J_A$. □

Remark 3 (Nonabelian positivity as equivariance)**.** For a larger group, for instance a semidirect product $K \rtimes \mathbb{R}$, positivity is not merely the sign choice in the time spectrum. The polar phase selected by the $\mathbb{R}$-subgroup must also satisfy an equivariance condition with respect to the orientation-preserving part of the remaining symmetry. Theorem 3 gives the basic criterion: symmetries preserving the oriented generator commute with J_A, while those reversing it implement the conjugate phase.

Once an orthogonal complex structure J is fixed, the usual complex Hilbert-space machinery is recovered.

Proposition 1 (Complex Hilbert reconstruction). *Let H be a real Hilbert space with orthogonal complex structure J. Define*

$$(a+ib)x = ax + bJx, \qquad a,b \in \mathbb{R}, \tag{23}$$

and define the complex inner product, linear in the first variable, by

$$\langle x, y\rangle_J = (x,y)_{\mathbb{R}} - i(Jx,y)_{\mathbb{R}}. \tag{24}$$

Then $(H, J, \langle\cdot,\cdot\rangle_J)$ is a complex Hilbert space. A bounded real-linear operator T is complex-linear for this structure if and only if $TJ = JT$.

Proof. The relations $J^2 = -\mathrm{Id}$ and $J^* = -J$ imply that multiplication by i is unitary and that (24) is Hermitian, positive, and complex-linear in the first variable. Completeness follows from the original real Hilbert norm. Finally, $T(ix) = T(Jx)$ equals $iT(x) = JT(x)$ exactly when $TJ = JT$. □

4.1. Composite systems and tensor products

The preceding construction is a one-particle, or single-real-representation, phase-selection result. It does not by itself assert that the raw real tensor product of two reconstructed systems is the standard composite quantum system. This distinction is essential. If (H_1, J_1) and (H_2, J_2) are real Hilbert spaces after phase selection, then the standard composite is the complex Hilbert tensor product

$$(H_1, J_1)\hat{\otimes}_{\mathbb{C}}(H_2, J_2), \tag{25}$$

not the unbalanced real tensor product $H_1 \hat{\otimes}_{\mathbb{R}} H_2$.

Equivalently, start from the algebraic real tensor product and impose the balancing relation

$$J_1 x \otimes y \sim x \otimes J_2 y, \qquad x \in H_1,\ y \in H_2. \tag{26}$$

The quotient is then completed in the Hilbert norm transported from the complex tensor product (25). The resulting real Hilbert space is canonically the underlying real Hilbert space of the usual complex composite, and the global imaginary unit is

$$J_{12}[x \otimes y] = [J_1 x \otimes y] = [x \otimes J_2 y]. \tag{27}$$

Thus the local selected phases must be compatible with a monoidal rule: the two copies of multiplication by i are identified by (26). This is precisely what fails if one uses $H_1 \hat{\otimes}_{\mathbb{R}} H_2$ with no further quotient or balance condition. In finite dimensions the difference is already visible dimensionally: if $\dim_{\mathbb{C}}(H_i, J_i) = n_i$, then the raw real tensor product has real dimension $4n_1n_2$, whereas the standard complex tensor product has real dimension $2n_1n_2$. The same phase-balanced compatibility was included in the broader framework [7]; it is restated here because the tensor-product issue is central to the present quantum-kinematic scope and to the recent real-versus-complex composition debate.

The J-balanced rule has the same structural purpose as the non-naive composition rules used in recent real-valued formulations: it removes the extra degrees of freedom of the raw real tensor product and identifies the two local actions of the imaginary unit [11,12]. In the present framework, however, the rule is imposed only after J_1 and J_2 have been selected, and it is exactly the usual scalar-balancing relation for the complex tensor product. It therefore recovers the underlying real Hilbert space of $(H_1, J_1)\hat{\otimes}_{\mathbb{C}}(H_2, J_2)$ rather than independently deriving a new real multipartite theory. Recent formulations instead take an appropriate modified or symplectic composition rule as a central postulate for establishing operational equivalence with complex quantum mechanics. The constructions are compatible in purpose but occur at different logical stages.

The conclusion is therefore deliberately limited. The Real-commutant theorem selects compatible local complex structures; the recovery of the standard entanglement-bearing composite uses the additional J-balanced tensor product rule. This places the present result in line with the recent real-number debate, where the composition law rather than the one-system real Hilbert space is the decisive issue [10–12]. A full monoidal reconstruction for arbitrary multipartite systems is beyond the scope of this letter.

Thus the selected bounded observable algebra is

$$B_J(H) = \{T \in B(H) : TJ = JT\}. \tag{28}$$

Physical bounded observables are the self-adjoint elements of this algebra; effects are positive contractions in this algebra; sharp events are its projections. Normal states are positive trace-class J-linear operators ρ with $\operatorname{tr}\rho = 1$, and probabilities are

$$p_\rho(E) = \operatorname{tr}(\rho E). \tag{29}$$

For vector states this is the usual Born rule, $p_\psi(E) = \langle\psi, E\psi\rangle_J$. In dimensions where the standard Gleason–von Neumann theorem applies, countably additive normal probability measures on the selected projection lattice are represented by density operators [24,25]. The present construction identifies the projection lattice to which these probability theorems are applied; it does not replace their hypotheses.

If, in addition, the selected real sector carries a symplectic form σ compatible with J, meaning

$$\sigma(Ju, Jv) = \sigma(u,v), \qquad \mu_J(u,v) = \sigma(u, Jv) \tag{30}$$

is symmetric and positive definite, with $\mu_J(u,u) > 0$ for $u \neq 0$, and with induced norm equivalent to the Hilbert norm, then the standard Weyl–CCR construction applies [4,26–28]. The one-particle complex Hilbert space has inner product

$$\langle u, v\rangle_h = \mu_J(u,v) - i\sigma(u,v). \tag{31}$$

For $f, g \in H$, using the Weyl convention

$$W(f)W(g) = \exp[-i\sigma(f,g)/2]\, W(f+g), \tag{32}$$

the complex structure J determines the pure quasifree state, for $f \in H$,

$$\omega_J(W(f)) = \exp\left(-\frac{1}{4}\|f\|_h^2\right). \tag{33}$$

If the time evolution is generated by $A = JB$ with $B = |A| \geq 0$, then B is the positive one-particle Hamiltonian and second quantization gives the nonnegative Fock Hamiltonian $d\Gamma(B)$.

Projection and records as a later layer. The reconstruction above stops at the selected noncommutative event algebra. A measurement interface should therefore be analyzed only after J has been fixed. This is the second side of the dual-time separation. The inner-generative time flow $U(t)$ supplies the reversible spectral orientation that selects J_A; record time is represented only by additional reduced or effective structures. In an enlarged reversible dynamics, eliminating inaccessible degrees of freedom gives exact causal memory; local dissipation requires an additional reduced-bath or Markov approximation [29–31]. Stable records, when produced by such a reduced dynamics, define an approximately diagonal observer-accessible algebra and justify operational collapse only as an effective record approximation, in the usual spirit of decoherence and einselection [32]. This phase-before-records separation is part of the dependency discipline of the present framework: records may test, stabilize, or coarse-grain the selected J-linear algebra, but they do not supply an independent imaginary unit. Quantitative record stability is not proved here.

5. Examples

The examples are intended as consistency checks and diagnostics for the abstract theorem. They do not claim new oscillator or Klein–Gordon physics. Their role is to show exactly where the familiar positive-frequency complex structure enters when the starting point is real.

5.1. Harmonic oscillator and the real energy space

Let $h_{\mathbb{R}}$ be a real Hilbert space and let $\omega \geq m\,\mathrm{Id} > 0$ be a positive self-adjoint operator on $h_{\mathbb{R}}$. The configuration variable is

$$q \in \mathrm{Dom}(\omega),$$

where $\mathrm{Dom}(\omega)$ is equipped with the graph-equivalent norm $\|q\|_\omega := \|\omega q\|$, and the momentum variable is

$$p \in h_{\mathbb{R}}.$$

The real energy Hilbert space is

$$E_{\mathbb{R}} := \mathrm{Dom}(\omega) \oplus h_{\mathbb{R}}, \qquad \|(q,p)\|_E^2 = \|\omega q\|^2 + \|p\|^2. \tag{34}$$

The first-order oscillator equation $\dot{q} = p$, $\dot{p} = -\omega^2 q$ has generator

$$A(q,p) = (p, -\omega^2 q), \qquad \mathrm{Dom}(A) = \mathrm{Dom}(\omega^2) \oplus \mathrm{Dom}(\omega), \tag{35}$$

viewed as an unbounded skew-adjoint operator on $E_{\mathbb{R}}$. Its strongly continuous orthogonal flow is

$$U(t)(q,p) = \big(\cos(t\omega)q + \omega^{-1}\sin(t\omega)p, -\omega\sin(t\omega)q + \cos(t\omega)p\big). \tag{36}$$

The isometric map

$$T : E_{\mathbb{R}} \to h_{\mathbb{R}} \oplus h_{\mathbb{R}}, \qquad T(q,p) = (\omega q, p), \tag{37}$$

conjugates the dynamics to

$$TU(t)T^{-1}(u,v) = \big(\cos(t\omega)u + \sin(t\omega)v, -\sin(t\omega)u + \cos(t\omega)v\big). \tag{38}$$

In the variables $(u,v) = T(q,p)$ the generator and positive factor are

$$\widetilde{A}(u,v) = (\omega v, -\omega u), \qquad \mathrm{Dom}(\widetilde{A}) = \mathrm{Dom}(\omega) \oplus \mathrm{Dom}(\omega), \tag{39}$$

$$|\widetilde{A}|(u,v) = (\omega u, \omega v). \tag{40}$$

Therefore $\widetilde{A} = \widetilde{J}|\widetilde{A}| = |\widetilde{A}|\widetilde{J}$ with

$$\widetilde{J}(u,v) = (v, -u). \tag{41}$$

Transporting back gives the canonical complex structure on $E_{\mathbb{R}}$:

$$J(q,p) = (\omega^{-1}p, -\omega q). \tag{42}$$

This is the standard oscillator complex structure, but here it appears as the polar phase of the real time-evolution generator on the explicitly defined energy space.

5.2. Stationary Klein–Gordon fields

Let $h_{\mathbb{R}}$ be the real Hilbert space of spatial square-integrable fields on a static Cauchy surface, and let ω^2 be the positive self-adjoint spatial Klein–Gordon operator after the usual energy-space reduction. On the regular sector where ω is strictly positive, Cauchy data are

$$(\varphi, \pi) \in E_{\mathbb{R}} = \mathrm{Dom}(\omega) \oplus h_{\mathbb{R}},$$

with the energy norm (34). The real time evolution is the oscillator flow (36), and the polar phase is therefore

$$J(\varphi,\pi) = (\omega^{-1}\pi, -\omega\varphi). \tag{43}$$

If ω has a kernel, the same formula holds on $(\ker\omega)^\perp$ with the reduced inverse $\omega^{-1}E((0,\infty))$, while the zero-mode sector is residual phase data in the sense of Section 3. In the usual complex mode variable

$$a = 2^{-1/2}(\omega\varphi + i\pi), \tag{44}$$

formula (43) is exactly multiplication by i on positive-frequency modes. Thus the familiar one-particle complex structure is recovered from the real generator rather than imposed as an initial scalar-field choice.

5.3. Compact inner-generative time

For a strongly continuous orthogonal representation of S^1, the nonzero Fourier modes occur in conjugate pairs $n, -n$. This is the compact model of inner-generative time: the group parameter is periodic, but its dual spectrum is still paired by canonical conjugation. Choosing the positive orientation of the circle selects the positive Fourier half. On the nonzero-mode sector this gives the compact analogue of the positive-frequency rule:

$$J = i\sum_{n>0}(E_n - E_{-n})\Big|_{H_{\mathbb{R}}}, \tag{45}$$

where E_n is the spectral projection of the complexified circle representation. Reversing the circle orientation sends n to $-n$ and changes J to $-J$, exactly as time reversal changes the polar phase in Theorem 3. The $n = 0$ sector is a zero-mode sector in the sense of Section 3 and carries only residual Real-commutant phase data. This example illustrates that the selection principle is not tied to the real line itself, but to an oriented one-parameter subgroup with paired nonzero spectrum.

5.4. A rotational obstruction

Let $n \geq 2$ and let $H_{\mathbb{R}} = L^2(\mathbb{R}^n, \mathbb{R})$ carry translations and the natural $O(n)$ action. Fourier transform reduces translation-invariant multiplicity-one complex structures to multipliers $i\sigma(\xi)$ with $\sigma(-\xi) = -\sigma(\xi)$. Full rotational invariance would require σ to be constant on spheres. Since $-I \in O(n)$ sends ξ to $-\xi$, invariance would give $\sigma(\xi) = \sigma(-\xi)$, contradicting oddness almost everywhere. Hence no bounded orthogonal complex structure commutes simultaneously with translations and the full $O(n)$ action in this scalar multiplicity-one setting. This example shows why selection must be tied to an oriented time subgroup rather than to arbitrary full symmetry.

6. Conclusion

The paper has isolated an operator-theoretic phase-selection layer inside real Hilbert representations. For a real orthogonal representation, the possible imaginary units are exactly the skew-adjoint square roots of $-\mathrm{Id}$ in the Real commutant of the complexified representation. A distinguished strongly continuous time flow then selects the physical complex structure on the nonzero-energy sector by

$$A = J_A|A| = |A|J_A.$$

This selection does not require a gap at zero. Zero modes are precisely the sector on which the oriented time flow does not determine a phase.

In dual-time language, the theorem identifies the exact Hilbert-space role of inner-generative time. The reversible group $U(t)$ supplies a real generator A; Stone duality turns it into a paired energy spectrum; positive orientation chooses one side of that pair; and the resulting polar phase is the operator that later plays the role of i. The irreversible or record-forming side of time belongs to projection, memory, decoherence, and measurement interfaces after this selected J-linear algebra exists.

The shared phase-selection core is best understood as an order of dependence rather than as a replacement for the standard foundations of quantum theory. Real symmetry classifies possible phases, inner-generative time orients the dual energy spectrum and selects one, and complex quantum kinematics is then reconstructed on the selected sector. The selected complex structure is covariant under symmetries preserving the time orientation and changes sign under symmetries reversing it. After this phase has been selected, the ordinary complex Hilbert space, J-linear observable algebra, normal-state probability calculus, and the Weyl–CCR/Fock representation under standard symplectic hypotheses follow by standard constructions. Relative to the broader framework [7], the present Letter's focused additions are the measurable type-I realization, the one-particle observable/probability/Weyl–

CCR development, and the explicit energy-space, compact-time, and symmetry-obstruction examples.

The reconstruction is deliberately limited. It does not claim that every real representation admits a global complex structure, that probability follows from no assumptions, or that measurement records are generated automatically. It also does not assert that the raw real tensor product gives the correct standard composite system. Composite systems require the J-balanced, equivalently complex, tensor product rule described in Section 4.1. In this sense the framework is a theory of phase selection for one-particle and operator-algebraic sectors, together with an explicit compatibility condition for standard composites, not a complete derivation of multipartite quantum mechanics from a bare real tensor product.

The limitations for quantum field theory are similarly explicit. Free or linear stationary models fit the one-particle theorem directly, and the Weyl–CCR/Fock construction applies once the usual symplectic and regularity hypotheses are supplied. Interacting quantum field theories require additional input: local nets, domains of interacting fields, renormalization, state-selection conditions such as spectrum or Hadamard requirements, and possibly modular localization. These may constrain the Real-commutant invariant, but they are not derived by the present one-particle phase-selection theorem.

Several extensions are now sharply formulated. The intrinsic Real-commutant phase space exists beyond type I, but type-II and type-III real von Neumann commutants require a separate structural analysis. Zero modes require independent residual phase data rather than being selected by positive energy. Local AQFT nets and modular localization may impose additional restrictions on the selected phase. Finally, projection, memory, decoherence, and stable records belong to a later reduced-dynamics layer. In dual-time terms, this later layer asks how irreversible record time acts on the phase already selected by inner-generative time: whether it preserves J_A, contracts residual multiplicity freedom, or selects a stationary/KMS-type state. Those questions are downstream of the present theorem. In this sense the complex formalism is not abandoned; it is recovered from real symmetry and positive energy before measurement or records enter the discussion.

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