

**CANONICAL COMPLEXIFICATION, MINIMAL
PROJECTION SECTORS, AND THE STRUCTURAL
INTEGER 137**

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ABSTRACT. This paper is a sequel-style structural application of canonical complexification. The imported input is the Real-commutant theorem: for an orthogonal representation U on a real Hilbert space, the algebra

$$A_{\mathbb{R}}(U) = \{T \in \mathcal{B}(H_{\mathbb{C}}) : TV(g) = V(g)T, TC = CT\}$$

classifies symmetry-compatible orthogonal complex structures as the skew-adjoint square roots of $-I$ in this real C^* -algebra; for a strongly continuous real orthogonal flow, positive energy selects a unique complex structure J on the nonzero-energy sector and gives the rigid factorization $A = JB = BJ$, $B \geq 0$.

Inside this selected complex phase structure we study finite projection sectors. A projection category P_n , branch-count functor B_n , and normalized valuation $W_{n,\lambda}$ determine

$$N_n(\lambda) = \sum_{d=0}^n (2^d - 1)\lambda^{n-d}, \quad N_3(\lambda) = \lambda^2 + 3\lambda + 7.$$

A minimal exact one-innovation tower in the positive-energy sector is a finite J -linear tower

$$0 \rightarrow K_d \xrightarrow{T_o} K_{d+1} \rightarrow \mathbb{C} \rightarrow 0, \quad d = 0, 1, 2,$$

with $\dim_J K_0 = 1$. Exactness gives $\dim_J K_d = d + 1$ and hence

$$\mathrm{Tr}_J(I_K) = 1 + 2 + 3 + 4 = 10.$$

Requiring the valuation scale to equal this complex trace gives $W_{3,10}$ and

$$N_3(\mathrm{Tr}_J(I_K)) = N_3(10) = 137.$$

Thus the integer 137 is a structural corollary of branch reconstruction plus an exact minimal projection tower after positive-energy canonical complexification.

Two extensions are recorded without entering the proof: quaternionic stability gives a possible route to the height-three case, and a closure hierarchy of multiplicity ten gives a minimal primitive one-slot readout of size ten. A final model-building section gives a cautious hydrogen interface. The standard Bohr–Sommerfeld identity says that the hydrogenic Coulomb speed ratio is $v/c = \alpha$; a separate re-creation lag ansatz writes a massive projected speed as $v/c = 1/N_{\mathrm{eff}}$. Identifying these ratios is a future matching hypothesis, not a theorem here. Likewise, the antiunitary reversal $J \mapsto -J$ defines a conjugate orientation sector; interpreting it as antihydrogen or a mirror sector requires additional charge-conjugation and gauge data. Any physical statement $\alpha^{-1} = 137 + \Delta$ requires a separate finite-regulator matching theorem with positivity, observable-sector selection, renormalization, and a computable correction term.

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1. INTRODUCTION

This paper develops a structural consequence of canonical complexification. In the canonical-complexification theorem for real Hilbert representations, the complex phase of a quantum sector is not assumed at the outset. Instead, for an orthogonal representation U on a real Hilbert space $H_{\mathbb{R}}$, one first forms the complexification $H_{\mathbb{C}}$, the complexified representation V , and the canonical conjugation C . The intrinsic controlling algebra is the Real commutant

$$A_{\mathbb{R}}(U) = \{T \in \mathcal{B}(H_{\mathbb{C}}) : TV(g) = V(g)T \text{ for all } g, TC = CT\}.$$

Symmetry-compatible orthogonal complex structures are exactly the skew-adjoint square roots of $-I$ in this real C^* -algebra. For a strongly continuous real orthogonal flow, positive energy then selects a unique element of this parameter space on the nonzero-energy sector: if $U(t) = e^{tA}$, then

$$A = JB = BJ, \quad B \geq 0,$$

and this positive-energy factorization is rigid. We use that result as the foundational phase-selection input of the present paper.

The question studied here is what finite projection sectors can be canonically normalized after this complex phase has already been selected. The

answer is not a derivation of a measured physical constant. It is a theorem about a finite algebraic-combinatorial and operator-theoretic normalization problem. Once the positive-energy complex structure J is fixed, a minimal exact one-innovation tower

$$0 \longrightarrow K_d \xrightarrow{T_o} K_{d+1} \longrightarrow \mathbb{C} \longrightarrow 0, \quad d = 0, 1, 2,$$

with $\dim_J K_0 = 1$ has complex dimensions

$$1, 2, 3, 4$$

by additivity of exact sequences. Its canonical trace scale is therefore

$$\lambda_{\text{op}} = \text{Tr}_J(I_K) = 1 + 2 + 3 + 4 = 10.$$

This trace scale is independent of a choice of coordinates in the selected complex sector; it is computed after positive-energy canonical complexification.

The combinatorial side is a projection reconstruction polynomial. A finite projection category P_n , a branch-count functor B_n , and a normalized multiplicative valuation $W_{n,\lambda}$ determine

$$(1) \quad N_n(\lambda) = \sum_{d=0}^n (2^d - 1) \lambda^{n-d}.$$

At height three,

$$(2) \quad N_3(\lambda) = \lambda^2 + 3\lambda + 7.$$

Thus the integer 137 is not an input. It is the value of the height-three branch-reconstruction polynomial at the canonical trace scale of the minimal exact tower:

$$N_3(\lambda_{\text{op}}) = N_3(10) = 137.$$

Equivalently, $\lambda = 10$ is the unique positive solution of $N_3(\lambda) = 137$.

The final theorem may be summarized as

positive-energy canonical complexification $\Rightarrow J$, exact one-innovation projection tower $\Rightarrow \text{Tr}_J(I_K) = 10$, height-three branch reconstruction $\Rightarrow N_3(10) = 137$.
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The statement is intentionally conditional but no longer treats the complex structure as a freely chosen datum. The complex phase is inherited from the Real-commutant/positive-energy theory, and the scale 10 is the complex trace of a minimal exact projection sector.

Two broader extensions are kept in the paper because they explain why the height-three and decadic features are natural targets, though they are not needed in the proof of $N_3(10) = 137$. First, a finite-dimensional associative norm-preserving nonabelian stability interface selects the quaternion algebra $\mathbb{H}$ by the Frobenius theorem, and the vector part $\text{Im } \mathbb{H}$ is three-dimensional. This supplies a possible structural route to the height-three case. Second,

any finite hierarchy of closure multiplicity ten has a unique minimal primitive one-slot readout of size ten. The standard quadratic closure flag

$$P_d = \dim \text{Sym}^2(E_d) = \binom{d+2}{2}, \quad d = 0, 1, 2, 3,$$

realizes this condition by the sequence 1, 3, 6, 10 and supplies a careful mathematical version of the claim that a decadic primitive scale may have structural roots.

Finally, the relation to the electromagnetic fine-structure constant is deliberately separated from the theorem. The observed inverse fine-structure constant is close to, but not equal to, 137. A physical matching theorem would have to construct a finite-regulator gauge model, identify a distinguished electromagnetic observable sector, prove positivity or reconstruction, control renormalization, and compute a correction term Δ in a formula of the schematic form

$$\alpha_{\text{phys}}^{-1} = 137 + \Delta.$$

The present paper supplies only the structural bare term and the mathematical slots in which such a correction could live.

A further model-building section connects this correction language to the simplest atomic system, but it is kept separate from the theorem. In the nonrelativistic Coulomb model of hydrogen, the ground-state Bohr–Sommerfeld relative speed satisfies $v/c = \alpha$. Independently, a re-creation lag ansatz may represent a massive projected excitation by an effective tick count N_{eff} , so that $v/c = 1/N_{\text{eff}}$ relative to a one-tick photon benchmark. A physical matching theorem would have to justify the identification

$$N_{\text{eff}} = \alpha_{\text{phys}}^{-1} = 137 + \Delta_{\text{phys}}.$$

Thus the structural value 137 is best read as a bare speed-denominator candidate. The correction Δ_{phys} must be produced by finite-regulator dynamics, radiative effects, reduced-mass effects, projection defects, and the choice of electromagnetic observable sector.

Claim-status discipline. We distinguish four kinds of statements.

- Definitions and assumptions specify the admissible class.
- Results labeled Lemma, Proposition, Theorem, Corollary, or Main Theorem are proved from those assumptions.
- Remarks and interpretations explain motivation or possible physical readings; they are not used as theorem-level inputs unless explicitly cited as assumptions.
- Theorem Targets identify analytic packages that a future finite-regulator gauge model would have to supply.

All uniqueness claims in the paper are relative to the admissible class under discussion. In particular, the theorem-level result is a conditional structural uniqueness theorem for a bare reconstruction value inside a positive-energy,

canonically complexified, exact one-innovation projection sector. It is not a derivation of the measured low-energy fine-structure constant.

Status and role of the hypotheses. Three inputs are especially important. First, the positive-energy complex structure J is imported from the canonical-complexification theorem and is not freely chosen in the present paper. Second, the exact one-innovation tower is a finite-sector hypothesis; once it exists, its dimension ladder is forced by exactness, but the existence of such a tower in a concrete regulator model remains a theorem target. Third, height three is the specialization used for the structural integer 137. The quaternionic stability filter supplies one independent mathematical route to that height, but the branch-reconstruction theorem itself is defined for every finite height.

The assumptions are minimal in a dependency sense. Canonical complexification supplies J ; exact one-dimensional cokernels supply the trace scale 10; the branch-count functor supplies the coefficients 0, 1, 3, 7; and the operator-admissibility condition identifies the valuation scale with the complex trace. Removing any one of these inputs leaves a formal freedom recorded in Proposition 5.4.

Main outputs. The paper proves or records, with explicit status, the following outputs.

- O1. The Real-commutant/positive-energy theory supplies a distinguished complex structure J on the nonzero-energy sector of a real orthogonal flow.
- O2. A finite projection category and branch-count functor determine the reconstruction polynomial (1).
- O3. At height three $N_3(\lambda) = \lambda^2 + 3\lambda + 7$. Once $\lambda = 10$ is selected, $N_3(10) = 137$ follows.
- O4. An exact J -linear one-innovation operator tower has complex dimensions $(1, 2, 3, 4)$ by additivity of exact sequences, and its complex trace scale is 10.
- O5. Therefore, within the operator-admissible class, the canonically complexified valuation is $W_{3,10}$ and the bare reconstruction value is $N_3(10) = 137$.
- O6. Under finite-dimensional associative, norm-preserving, nonabelian stable-composition assumptions, the quaternionic envelope $\mathbb{H}$ is selected and its vector part is three-dimensional. This is a possible height-three selection mechanism.
- O7. For any finite hierarchy with closure multiplicity M , the minimal locally faithful primitive alphabet has size M . Hence any height-three hierarchy with $M = 10$ has a unique minimal primitive readout scale equal to ten.
- O8. The triangular sequence 1, 3, 6, 10 is derived as the quadratic closure count $\dim \text{Sym}^2(E_d)$ for a rank- $(d+1)$ flag. It is a canonical realization of the decadic condition, not an independent universal law.

- O9. The sexagesimal and circular subdivision pattern $10 \cdot 6 \cdot 6 = 360$ is recorded as a compatible interpretation of the quadratic closure hierarchy; it is not part of the proof of 137.
- O10. The hydrogen extension records a possible matching interface: the Coulomb/Bohr speed identity gives $v/c = \alpha$, while a re-creation lag ansatz gives $v/c = 1/N_{\text{eff}}$. Equating these ratios is a physical matching hypothesis, not a theorem of the structural core.
- O11. A conjugate orientation sector may be formulated by antiunitary reversal of the selected complex structure. The bare trace scale is invariant under such conjugation; identifying this sector with physical antihydrogen or a mirror world requires additional charge-conjugation and gauge-sector input.
- O12. Correction terms, reflection positivity, transfer-matrix reconstruction, finite-regulator physical matching, and measured coupling values remain downstream theorem targets.

2. CANONICAL-COMPLEXIFICATION INPUT

This section records the part of canonical complexification used as an input in the present paper. The statements are included to keep the sequel logically readable; the full Real-commutant classification, measurable type-I computation, and quasifree/Fock consequences are treated in [1].

Let $U : G \rightarrow O(H_{\mathbb{R}})$ be an orthogonal representation on a real Hilbert space. Let $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification, let V be the complexified unitary representation, and let C denote canonical conjugation.

Definition 2.1 (Real commutant). The Real commutant of U is

$$A_{\mathbb{R}}(U) = \{T \in \mathcal{B}(H_{\mathbb{C}}) : TV(g) = V(g)T \text{ for all } g \in G, TC = CT\}.$$

The symmetry-compatible orthogonal complex structures on $H_{\mathbb{R}}$ are

$$\mathcal{X}_{\mathbb{R}}(U) = \{J \in \mathcal{B}(H_{\mathbb{R}}) : JU(g) = U(g)J, J^* = -J, J^2 = -I\}.$$

Theorem 2.2 (Canonical-complexification theorem, imported). *For every orthogonal representation U , complexification identifies $\mathcal{X}_{\mathbb{R}}(U)$ with the set of skew-adjoint square roots of $-I$ in the real C^* -algebra $A_{\mathbb{R}}(U)$. If $U(t) = e^{tA}$ is a strongly continuous orthogonal flow and the nonzero-energy sector is oriented by positive energy, then there is a unique compatible complex structure J on that sector such that*

$$A = JB = BJ, \quad B \geq 0.$$

This positive-energy factorization characterizes J inside the positive-energy factorization class.

Interpretation 2.3 (Role in the present paper). Theorem 2.2 removes the main ambiguity of the operator layer. The complex structure used to count dimensions and traces in the finite projection sector is not an arbitrary scalar i attached after the fact. It is the positive-energy point selected in the Real-commutant parameter space of the underlying real Hilbert dynamics.

Definition 2.4 (Post-complexification finite sector). A post-complexification finite sector is a finite-dimensional J -invariant real subspace K of the positive-energy dynamical sector selected by Theorem 2.2. It is counted as a complex vector space using the selected operator J .

3. DIMENSIONALITY SELECTION: THE THREE-DIMENSIONAL STABILITY FILTER

The reconstruction polynomial is defined for every finite height n , so a separate question must be answered before the three-dimensional specialization can be treated as structurally privileged. This section records the dimensionality filter used in the merged framework. The filter is independent of the numerical evaluation $N_3(10) = 137$: it says why the stable projection stage relevant to ordinary spatial reconstruction is the quaternionic, three-vector stage rather than an arbitrary member of the family P_n .

The status of this section is the following. The algebraic statements are theorem-level consequences of the stated stability assumptions together with classical classification results for finite-dimensional real associative division algebras and normed division algebras. Any physical vocabulary attached to this section is interpretive. The theorem-level statement uses only the stable-composition interface: update phases compose, coherent magnitudes multiply, and only directions closed under repeated composition survive in the stable envelope.

3.1. Minimal operational interface. The following assumptions are the small operational core needed for the dimensionality argument. They deliberately avoid relying on a pre-existing background spacetime metric.

Assumption 3.1 (Stable projection-composition interface). A stable inner-time projection system supplies the following data.

- (i) A finite-dimensional real effective mode space A generated by coarse-grained inner-time update phases.
- (ii) A bilinear multiplication $A \times A \rightarrow A$, denoted $(x, y) \mapsto xy$, representing composition of stable update phases.
- (iii) A unit element $1 \in A$, representing the neutral completed update.
- (iv) A positive-definite norm $\| \cdot \|$ representing projection coherence.
- (v) Multiplicative coherence:

$$\|xy\| = \|x\| \|y\|.$$

- (vi) Associativity:

$$(xy)z = x(yz),$$

so that long histories have unambiguous coarse composition.

- (vii) Genuine rotational closure: the stable update algebra is not required to be commutative; noncommuting unit elements are permitted and encode order-sensitive rotations.

The finite-dimensionality condition is an operational finite-information condition: at fixed outer resolution, a bounded region has finitely many distinguishable stable modes. The multiplicative norm says that composing two coherent updates does not create or destroy coherence magnitude except by the product of the input magnitudes. Associativity is the history-independence condition: if three updates are composed, the final coarse update cannot depend on how the composition is parenthesized.

Definition 3.2 (Stable composition algebra). A stable composition algebra is a tuple

$$(A, \cdot, 1, \langle \cdot, \cdot \rangle)$$

where A is a finite-dimensional real unital associative algebra and $\langle \cdot, \cdot \rangle$ is a positive-definite inner product whose norm satisfies $\|xy\| = \|x\|\|y\|$ for all $x, y \in A$. The vector part of A is

$$A_{\text{vec}} := \{a \in A : \langle a, 1 \rangle = 0\}.$$

Lemma 3.3 (Multiplicative norm implies division). *Let A be a stable composition algebra. If $a \neq 0$, then left multiplication $L_a(x) = ax$ and right multiplication $R_a(x) = xa$ are injective. Since A is finite-dimensional, they are bijective. Consequently every nonzero element of A has a left and right inverse, and A is a finite-dimensional real associative division algebra.*

Proof. If $a \neq 0$ and $ax = 0$, then

$$0 = \|ax\| = \|a\|\|x\|.$$

Since $\|a\| > 0$, it follows that $x = 0$. Thus L_a is injective, hence bijective in finite dimension. Similarly R_a is bijective. Bijectivity gives elements b, c with $ba = 1$ and $ac = 1$. Then $b = b(ac) = (ba)c = c$, so the left and right inverses coincide. $\square$

Theorem 3.4 (Associative division-algebra filter). *Every finite-dimensional real associative division algebra is isomorphic to one of*

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}.$$

Therefore, if stable projection composition is required to be noncommutative, the only associative possibility is the quaternion algebra $\mathbb{H}$.

Proof. This is the Frobenius classification theorem for finite-dimensional associative real division algebras. The algebras $\mathbb{R}$ and $\mathbb{C}$ are commutative; $\mathbb{H}$ is the unique noncommutative case. $\square$

Corollary 3.5 (Quaternionic vector part is three-dimensional). *If the stable projection-composition algebra is noncommutative, associative, finite-dimensional, and norm-preserving, then*

$$A \cong \mathbb{H} = \mathbb{R} \oplus \text{Im } \mathbb{H},$$

where

$$\text{Im } \mathbb{H} = \text{Span}_{\mathbb{R}}\{i, j, k\}$$

has real dimension three. Unit quaternions form $S^3 \cong \mathrm{SU}(2)$ and act on $\mathrm{Im} \mathbb{H} \cong \mathbb{R}^3$ by

$$R_q(v) = qvq^{-1}, \quad \|q\| = 1,$$

giving the spin covering map $\mathrm{SU}(2) \rightarrow \mathrm{SO}(3)$.

Proof. The decomposition $\mathbb{H} = \mathbb{R} \oplus \mathrm{Span}_{\mathbb{R}}\{i, j, k\}$ is the standard scalar-vector decomposition of quaternions. The multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = k, jk = i, \quad ki = j$$

with antisymmetry under exchange generate a noncommutative associative normed algebra. Conjugation by unit quaternions preserves the pure-imaginary subspace and its Euclidean norm, and the resulting action is the standard double cover of $\mathrm{SO}(3)$. $\square$

3.2. The minimal associative stability theorem. The preceding classification turns the qualitative statement “stable rotations need three directions” into a precise algebraic statement.

Theorem 3.6 (Minimal associative projection stability). *Under Assumption 3.1, if stable projection composition is required to be norm-preserving, associative, and genuinely nonabelian, then the minimal nontrivial spatial vector sector is three-dimensional. It is realized by*

$$\mathrm{Im} \mathbb{H} = \mathrm{Span}_{\mathbb{R}}\{i, j, k\}.$$

No smaller associative case supports nonabelian rotational closure, and any attempt to move to the next normed division algebraic envelope leads to the octonionic case, which is nonassociative.

Proof. By Lemma 3.3, a stable finite-dimensional associative normed composition algebra is a real associative division algebra. By Theorem 3.4, the only possibilities are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The cases $\mathbb{R}$ and $\mathbb{C}$ are commutative. They can encode scalar or planar phase behavior but not nonabelian rotational closure. The quaternion algebra $\mathbb{H}$ is the first and only noncommutative associative case; its vector part has dimension three by Corollary 3.5. If associativity is dropped, the Hurwitz list of normed division algebras also includes the octonions $\mathbb{O}$, but $\mathbb{O}$ is nonassociative, so composite update histories would depend on parenthesization. That violates the associativity clause of stable projection composition. $\square$

Corollary 3.7 (Uniqueness of the three projection axes). *Any stable projection scheme satisfying the hypotheses of Theorem 3.6 has exactly three independent noncommuting vector generators up to algebra isomorphism. These may be represented by the three imaginary quaternion units i, j, k , or by three skew real 4×4 matrices satisfying the same quaternionic relations.*

Interpretation 3.8 (Projection reading). In a projection-history reading, the first three stable projection directions are identified with the three quaternionic vector generators. The statement is not that an observer arbitrarily

chooses three axes, but that an associative, norm-preserving, nonabelian update envelope has exactly three independent vector directions.

3.3. Why the alternatives fail. The three-dimensional result can be summarized by considering the possible numbers of independent projection directions.

Case	Algebraic behavior	Result for stable spatial projection
$n < 3$	Real or complex/planar behavior; composition is commutative or effectively abelian.	Insufficient for full nonabelian rotational closure and spinorial double cover.
$n = 3$	Quaternionic envelope $\mathbb{H}$; associative, norm-preserving, noncommutative.	Stable vector part $\text{Im } \mathbb{H} \cong \mathbb{R}^3$ with unit-quaternion rotations $\text{SU}(2) \rightarrow \text{SO}(3)$.
$n > 3$	The next normed division-algebraic enlargement is octonionic.	Nonassociativity creates ambiguous composite projection histories; extra directions are either redundant or incoherent under the associative stability requirement.

This table is the dimensional counterpart of the later decimal and 137 uniqueness tables. Each uniqueness statement excludes alternatives by specifying which axiom an alternative violates.

3.4. Topological universality route. There is a second, more global, route to the same selection. It is useful because it shows that the quaternionic result is not merely an artifact of beginning with algebras. Suppose the neutral stable update sector has a compact Lie closure K , an isotropy subgroup H , and a homogeneous stable-envelope space $X = K/H$. If coarse reblocking supplies multiplication on X up to homotopy, then X is an H -space. Under the usual two-point-homogeneity or rank-one symmetry assumptions, the stable envelope is spherical.

Theorem 3.9 (Spherical H -space restriction). *If a spherical stable envelope $X \simeq S^m$ admits an H -space multiplication, then*

$$m \in \{0, 1, 3, 7\}.$$

If the coarse composition is required to be associative and noncommutative in the stable finite-dimensional envelope, the admissible nonabelian associative case is the quaternionic one, corresponding to $m = 3$ and envelope $S^3 \cong \text{SU}(2)$.

Proof. The restriction $m \in \{0, 1, 3, 7\}$ is the Adams Hopf-invariant-one theorem in its standard H -space sphere form. The cases correspond to unit spheres in $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. The octonionic case S^7 is nonassociative at the algebraic envelope level. The real and complex cases are commutative or

abelian for the relevant rotational closure. Hence the unique associative noncommutative stable envelope is quaternionic. $\square$

Remark 3.10 (Envelope sphere versus ordinary direction sphere). The sphere S^3 in Theorem 3.9 is the unit-quaternion envelope, not the ordinary sphere of directions in $\mathbb{R}^3$. The latter is S^2 . The physical three-vector space is the tangent/pure-imaginary sector $\text{Im } \mathbb{H}$, while $S^3 \cong \text{SU}(2)$ is the multiplicative spin envelope acting on it.

3.5. Dynamical diagnostics and rotational diffusion. The quaternionic selection can also be probed by toy stochastic dynamics. Let J_1, J_2, J_3 be real skew generators satisfying quaternionic relations. A microscopic update may be modeled schematically by

$$X_m = \exp\left(\varepsilon \sum_{a=1}^3 \xi_{m,a} J_a\right),$$

where the random coefficients have mean zero and finite covariance. Products $X_m \cdots X_1$ live in an $\text{SU}(2)$ envelope. In a diffusive scaling limit, such random products converge to Brownian motion on $\text{SU}(2)$ with a rotational diffusion constant D determined by the microscopic variance and tick scale.

At the observable level this produces a low-frequency angular random-walk spectrum of the schematic form

$$S_\theta(f) \simeq \frac{2D}{(2\pi f)^2}.$$

The present paper does not use this prediction in the proof of 137. Its role is methodological: a dimensionality-selection mechanism should not remain purely verbal. A concrete micro-update model must produce bounded, testable diffusion or be excluded by precision rotation and phase-noise experiments.

Theorem Target 3.11 (Rotational-diffusion diagnostic). For a specified finite-regulator update model, prove a random-product limit theorem on $\text{SU}(2)$, derive the exact relation between the microscopic tick variance and D , and compare the predicted $S_\theta(f)$ or Allan-variance scaling with experimental bounds.

3.6. Use of the three-dimensional filter in the rest of the paper. The remainder of the paper does not need to reprove quaternionic dimensionality at every step. It uses the filter in one precise way: the admissible reconstruction stage is $n = 3$. Once this stage has been selected, two independent ten-counts enter:

$$P_3 = 10 \quad \text{and} \quad \dim_J K = 10.$$

The first is a minimal-readout count for a closure hierarchy of maximal multiplicity ten; the quadratic realization gives $P_3 = 10$. The second is the operator trace scale of the canonically complexified exact rank-one tower. The reconstruction polynomial then uses the branch-count ladder $0, 1, 3, 7$,

not the quadratic closure ladder 1, 3, 6, 10. Keeping these ladders separate is essential.

4. DECADIC SCALE SELECTION: PRIMITIVE READOUT AND QUADRATIC CLOSURE

This section separates two questions that were conflated in earlier formulations. The first is a general readout problem: how large must a primitive alphabet be in order to label a maximal layer in one slot? The second is a closure problem: which finite hierarchy supplies a maximal layer of size ten? The main theorem only needs the answer to the second question to be $M = 10$. The triangular sequence 1, 3, 6, 10 is supplied here as a canonical quadratic realization of that decadic condition.

4.1. Closure multiplicity and primitive readout.

Definition 4.1 (Finite graded multiplicity hierarchy). A finite graded multiplicity hierarchy is a sequence

$$P = (P_0, P_1, \dots, P_{d_*}), \quad P_d \in \mathbb{N},$$

where P_d is the number of distinguishable projection elements at level d . Its closure multiplicity is

$$M(P) = \max_{0 \leq d \leq d_*} P_d.$$

Definition 4.2 (Primitive symbol inventory and locally faithful readout). A primitive symbol inventory is a finite set

$$\Sigma = \{\sigma_0, \dots, \sigma_{b-1}\}$$

whose elements are treated as elementary, one-slot symbols rather than as strings in a smaller alphabet. A locally faithful elementary readout at a level with M elements is an injective assignment from those M elements into Σ .

Proposition 4.3 (Minimal elementary readout). *For a hierarchy with closure multiplicity M , any locally faithful elementary readout requires*

$$b \geq M.$$

Conversely, $b = M$ is sufficient. Thus the unique minimal primitive alphabet size is

$$b_{\min} = M.$$

Proof. An injective assignment from an M -element set into Σ exists only if $|\Sigma| = b \geq M$. If $b = M$, choose any bijection between the maximal-level elements and the symbols in Σ . $\square$

Remark 4.4 (Composite encodings). Proposition 4.3 does not privilege decimal notation for arbitrary computation. It distinguishes primitive one-slot readout from composite encoding. An M -element set can be encoded by binary strings, but that representation is not an elementary one-symbol labeling of the maximal layer.

4.2. Quadratic closure flags.

Definition 4.5 (Quadratic closure flag). A height-three quadratic closure flag consists of finite-dimensional vector spaces

$$E_0 \subset E_1 \subset E_2 \subset E_3, \quad \dim E_d = d + 1,$$

over $\mathbb{R}$ or $\mathbb{C}$. The level- d closure elements are the symmetric quadratic generators of E_d , so the multiplicity at level d is

$$P_d := \dim \operatorname{Sym}^2(E_d).$$

Proposition 4.6 (Quadratic closure counts). *A quadratic closure flag has multiplicities*

$$P_d = \dim \operatorname{Sym}^2(E_d) = \binom{d+2}{2} = \frac{(d+1)(d+2)}{2}, \quad d = 0, 1, 2, 3.$$

In particular,

$$(P_0, P_1, P_2, P_3) = (1, 3, 6, 10).$$

Proof. If $\dim E_d = d + 1$, then the dimension of the symmetric square is the number of unordered pairs with repetition drawn from $d + 1$ basis directions:

$$\dim \operatorname{Sym}^2(E_d) = \binom{(d+1) + 1}{2} = \binom{d+2}{2}.$$

Evaluating at $d = 0, 1, 2, 3$ gives 1, 3, 6, 10. $\square$

Remark 4.7 (Why this weakens the hierarchy assumption). The main theorem below does not require a classification of all height-three hierarchies. It requires only a hierarchy with closure multiplicity ten. Proposition 4.6 supplies a canonical source of such a hierarchy from a standard functor, $E \mapsto \operatorname{Sym}^2(E)$.

4.3. The decadic readout theorem.

Theorem 4.8 (Decadic primitive readout). *Let P be any finite graded multiplicity hierarchy with closure multiplicity*

$$M(P) = 10.$$

If readout is required to be locally faithful and elementary in the sense of Definition 4.2, then the unique minimal primitive scale is

$$b_{\min} = 10.$$

Every smaller primitive alphabet loses one-slot local faithfulness; every larger primitive alphabet is nonminimal.

Proof. This is Proposition 4.3 with $M = 10$. $\square$

Corollary 4.9 (Quadratic realization of the decadic scale). *For the height-three quadratic closure flag of Definition 4.5, the unique minimal locally faithful primitive alphabet has ten symbols.*

Proof. By Proposition 4.6, the closure multiplicity is $P_3 = 10$. Apply Theorem 4.8. $\square$

4.4. From a ten-symbol alphabet to positional notation. Once a primitive alphabet Σ has b elements, the ordinary positional extension uses radix b :

$$N = \sum_{k=0}^m a_k b^k, \quad a_k \in \Sigma.$$

For $b = 10$ this is decimal notation. The theorem-level claim is not that all agents must use decimal notation, but that a hierarchy with $M = 10$ has a unique minimal primitive one-slot readout scale equal to ten.

4.5. Four convergent ten-counts. The merged framework contains several appearances of 10. They should not be identified.

Source of ten	Mechanism	Status
Closure multiplicity	$M(P) = 10$; for the quadratic flag, $P_3 = \binom{5}{2} = 10$	Readout theorem
Geometric accumulation	$0 + 1 + 3 + 6 = 10$	Geometric motivation
Operator tower	$1 + 2 + 3 + 4 = 10$ from exact rank-one extensions	Operator theorem
Sexagesimal subdivision	$10 \cdot 6 \cdot 6 = 360$	Interpretive compatibility

The first ten-count explains primitive readout. The third selects the valuation scale. The second and fourth are explanatory and are not used to prove $N_3(10) = 137$.

4.6. Sexagesimal and the 360-fold circle. For the quadratic closure flag one has $P_3 = 10$ and $P_2 = 6$. Applying two surface-level refinements to ten primary sectors gives

$$P_3 P_2 P_2 = 10 \cdot 6 \cdot 6 = 360.$$

This compatibility explains how a decadic primitive scale and a sexagesimal circular subdivision can coexist in the same hierarchy. It is not a historical derivation of sexagesimal notation and is not used in the proof of the structural integer 137.

4.7. Constrained information cost. Let $C(r)$ be any strictly increasing cost of primitive alphabet size r . If the local-faithfulness constraint imposes $r \geq 10$, then

$$\min_{r \in \mathbb{N}, r \geq 10} C(r) = C(10).$$

Thus cost minimization selects ten only after the readout constraint supplies the admissible range.

5. THE POST-COMPLEXIFICATION STRUCTURAL THEOREM

We now collect the hypotheses used by the canonical-complexification, branch-reconstruction, and projection-tower filters. The theorem is stated for an abstract package. No physical interpretation is needed for the proof.

Definition 5.1 (Post-complexification projection package). A post-complexification projection package consists of the following data.

- (H1) **Positive-energy real-Hilbert input.** A real Hilbert representation or real orthogonal flow whose nonzero-energy sector carries the positive-energy complex structure J supplied by Theorem 2.2.
- (H2) **Height-three branch reconstruction.** The height-three projection category P_3 equipped with the branch-count functor B_3 and a normalized multiplicative valuation $W_{3,\lambda}$.
- (H3) **Exact one-innovation projection tower.** In the selected positive-energy complex sector there is a finite-dimensional J -linear tower

$$K = \bigoplus_{d=0}^3 K_d$$

satisfying the exact rank-one conditions of Definition 12.1.

- (H4) **Operator-admissible scale.** The valuation scale agrees with the complex trace scale of this tower:

$$\lambda = \lambda_{\text{op}} = \text{Tr}_J(I_K).$$

If, in addition, one wants a height-three selection mechanism and a primitive-readout interpretation, one may add the following auxiliary data.

- (A1) **Quaternionic height selection.** A stable nonabelian associative norm-preserving finite-dimensional composition envelope as in Assumption 3.1.
- (A2) **Decadic closure/readout.** A finite graded multiplicity hierarchy $P = (P_0, P_1, P_2, P_3)$ with $M(P) = 10$, together with locally faithful primitive one-slot readout.

The main structural value 137 uses (H1)–(H4). The auxiliary assumptions (A1)–(A2) explain why height three and a decadic primitive scale are natural extensions, but they are not needed to evaluate $N_3(10)$ once the height-three tower is given.

Main Theorem 5.2 (Structural integer from a canonically complexified projection tower). *Let X be a post-complexification projection package satisfying Definition 5.1(H1)–(H4). Then:*

- (a) *the exact tower has dimension ladder*

$$\dim_J K_d = d + 1, \quad d = 0, 1, 2, 3,$$

and trace scale

$$\lambda_{\text{op}} = \text{Tr}_J(I_K) = 1 + 2 + 3 + 4 = 10;$$

- (b) *the operator-admissible valuation is $W_{3,10}$;*
- (c) *the branch-count reconstruction polynomial satisfies*

$$N_3(\lambda_{\text{op}}) = N_3(10) = 137;$$

- (d) *$\lambda = 10$ is the unique positive solution of $N_3(\lambda) = 137$.*

Thus, inside the post-complexification projection class, the integer 137 is fixed by the selected positive-energy complex structure, exact one-innovation trace scale, and height-three branch reconstruction.

Proof. The positive-energy real-Hilbert input supplies the complex structure J by Theorem 2.2. The exact tower gives the dimension ladder by Theorem 12.4. Hence

$$\lambda_{\text{op}} = \text{Tr}_J(I_K) = \dim_J K = 1 + 2 + 3 + 4 = 10.$$

Operator-admissibility therefore gives $\lambda = 10$ and the valuation $W_{3,10}$. The reconstruction theorem, Theorem 6.8, gives

$$N_n(\lambda) = \sum_{d=0}^n (2^d - 1) \lambda^{n-d},$$

and Corollary 6.11 gives $N_3(\lambda) = \lambda^2 + 3\lambda + 7$. Substituting $\lambda = 10$ gives

$$N_3(10) = 100 + 30 + 7 = 137.$$

Finally, Corollary 6.12 gives uniqueness of the positive solution. $\square$

Corollary 5.3 (Optional height-three and decadic extensions). *If the package also satisfies the auxiliary assumptions in Definition 5.1, then:*

(i) *the stable composition envelope is quaternionic,*

$$A \cong \mathbb{H}, \quad A_{\text{vec}} \cong \text{Im } \mathbb{H} \cong \mathbb{R}^3,$$

so the selected vector stage is three-dimensional;

(ii) *the minimal locally faithful primitive readout of the closure hierarchy has cardinality*

$$b_{\min} = 10.$$

Proof. Part (i) is Theorem 3.6. Part (ii) is Theorem 4.8. $\square$

Proposition 5.4 (Nonredundancy of the hypotheses). *Each structural hypothesis has an independent role.*

- (a) *Without positive-energy canonical complexification, the complex trace $\text{Tr}_J(I_K)$ is not intrinsically selected.*
- (b) *Without the exact one-innovation tower, the trace scale need not be 10.*
- (c) *Without the branch-count functor, the coefficients 0, 1, 3, 7 and hence the polynomial N_3 are not fixed.*
- (d) *Without operator-admissibility of the valuation scale, normalized valuations $W_{3,\lambda}$ with arbitrary $\lambda > 0$ remain available.*
- (e) *Without quaternionic stability, height three is not selected by the algebraic stability filter.*
- (f) *Without decadic closure and primitive one-slot readout, the number ten has no primitive-readout interpretation.*

Proof. Deleting the indicated hypothesis leaves the corresponding formal freedom. In (a), one can choose different compatible complex structures when the Real-commutant parameter space is not oriented by positive energy. In (b), finite J -invariant sectors can have arbitrary complex dimension. In (c), no subset-count coefficients have been specified. In (d), Proposition 7.6 leaves a free transitive $\mathbb{R}_{>0}$ -scale action on normalized valuations. In (e), $\mathbb{R}$

and $\mathbb{C}$ are abelian while the octonionic enlargement is nonassociative. In (f), finite sets admit composite encodings and arbitrary closure multiplicities. $\square$

Remark 5.5 (The three ladders). The theorem uses three different finite ladders. The branch ladder $0, 1, 3, 7$ supplies the coefficients of N_3 . A closure hierarchy with maximum 10 supplies the primitive readout scale; the quadratic realization gives $1, 3, 6, 10$. The operator ladder $1, 2, 3, 4$ supplies the trace scale. The integer 137 appears only when the branch-count polynomial is evaluated at the independently selected operator ten-scale.

6. PROJECTION CATEGORIES AND RECONSTRUCTION POLYNOMIALS

6.1. The finite projection category.

Definition 6.1 (Projection category). For $n \in \mathbb{N}$, let P_n be the finite poset category whose objects are

$$L_0, L_1, \dots, L_n,$$

and with a unique morphism $L_d \rightarrow L_e$ precisely when $d \leq e$. Equivalently, P_n is the thin category associated with the totally ordered set $0 < 1 < \dots < n$. The object L_d is called the d th projection layer.

This category records only the order structure of a reconstruction hierarchy. No physical interpretation is imposed at this stage.

6.2. Branch counting and the binary coefficient.

Definition 6.2 (Branch-count functor). Fix $n \in \mathbb{N}$. For $d \in \{0, 1, \dots, n\}$ let

$$[d] = \{1, 2, \dots, d\}, \quad [0] = \emptyset.$$

Define a functor

$$B_n : P_n \longrightarrow \mathbf{FinSet}$$

by

$$B_n(L_d) = \{S \subseteq [d] : S \neq \emptyset\}.$$

For the unique morphism $L_d \rightarrow L_e$ with $d \leq e$, define $B_n(L_d \rightarrow L_e)(S) = S$, viewing $S \subseteq [d]$ canonically as a subset of $[e]$.

Proposition 6.3 (Cardinality). For $0 \leq d \leq n$,

$$|B_n(L_d)| = 2^d - 1.$$

Proof. The set $[d]$ has 2^d subsets. Removing the empty subset leaves $2^d - 1$ nonempty subsets. $\square$

Remark 6.4 (Correction of the coefficient). The theorem-level coefficient is the exponential quantity $2^d - 1$, not $2d - 1$. It is forced by the subset count, by the recurrence below, and by the specialization $N_3(\lambda) = \lambda^2 + 3\lambda + 7$.

6.3. Multiplicative valuations and the polynomial.

Definition 6.5 (Normalized multiplicative valuation). Fix $n \in \mathbb{N}$ and $\lambda > 0$. A normalized multiplicative valuation on P_n is a function

$$W_{n,\lambda} : \text{Ob}(P_n) \longrightarrow \mathbb{R}_{>0}$$

such that

$$W_{n,\lambda}(L_n) = 1, \quad W_{n,\lambda}(L_{d-1}) = \lambda W_{n,\lambda}(L_d)$$

for $1 \leq d \leq n$.

Proposition 6.6 (Explicit form). For $0 \leq d \leq n$,

$$W_{n,\lambda}(L_d) = \lambda^{n-d}.$$

Proof. Start from $W_{n,\lambda}(L_n) = 1$ and iterate the adjacent-layer ratio condition. $\square$

Definition 6.7 (Reconstruction polynomial). For $n \in \mathbb{N}$ and $\lambda > 0$, define

$$N_n(\lambda) = \sum_{d=0}^n |B_n(L_d)| W_{n,\lambda}(L_d).$$

Theorem 6.8 (General reconstruction theorem). For every $n \in \mathbb{N}$ and $\lambda > 0$,

$$(3) \quad N_n(\lambda) = \sum_{d=0}^n (2^d - 1) \lambda^{n-d}.$$

Proof. Substitute the two preceding propositions into the definition of N_n . $\square$

Corollary 6.9 (Recurrence). The polynomials satisfy

$$N_n(\lambda) = \lambda N_{n-1}(\lambda) + (2^n - 1), \quad N_0(\lambda) = 0.$$

Proof. Separate the term $d = n$ in (3). $\square$

Corollary 6.10 (Closed form). For $\lambda \neq 1, 2$,

$$(4) \quad N_n(\lambda) = \frac{\lambda^{n+1} - 2^{n+1}}{\lambda - 2} - \frac{\lambda^{n+1} - 1}{\lambda - 1}.$$

At the exceptional values,

$$N_n(1) = 2^{n+1} - n - 2, \quad N_n(2) = (n - 1)2^n + 1.$$

Proof. Split the defining sum into

$$\sum_{d=0}^n 2^d \lambda^{n-d} - \sum_{d=0}^n \lambda^{n-d}$$

and evaluate the two geometric sums. The exceptional values follow by direct summation or by taking limits. $\square$

Corollary 6.11 (Three-dimensional specialization). In dimension three,

$$N_3(\lambda) = \lambda^2 + 3\lambda + 7.$$

Proof. Evaluate (3) at $n = 3$:

$$N_3(\lambda) = 0 \cdot \lambda^3 + 1 \cdot \lambda^2 + 3\lambda + 7.$$

□

Corollary 6.12 (Unique positive specialization to 137). *The equation*

$$N_3(\lambda) = 137$$

has the unique positive solution $\lambda = 10$.

Proof. The equation is $\lambda^2 + 3\lambda + 7 = 137$, or $\lambda^2 + 3\lambda - 130 = 0$. The roots are 10 and -13 , so the unique positive root is 10. □

Remark 6.13 (What is determined). The theorem-level object is the polynomial $N_3(\lambda)$, not the raw integer 137. The integer 137 appears only after a positive scale specialization. The operator part of the paper is devoted to making that scale specialization intrinsic inside a stated admissible class.

6.4. Three-dimensional bookkeeping.

d	$2^d - 1$	$W_{3,\lambda}(L_d)$	Contribution	$\lambda = 10$
0	0	λ^3	0	0
1	1	λ^2	λ^2	100
2	3	λ	3λ	30
3	7	1	7	7
Total			$\lambda^2 + 3\lambda + 7$	137

7. BRANCH LADDERS, GEOMETRIC LADDERS, AND VALUATION GAUGE

7.1. Branch count versus geometric direction count. The branch ladder

$$0 : 1 : 3 : 7$$

counts nonempty subsets of a d -element axis set. A different sequence sometimes appears in geometric interpretations:

$$0 : 1 : 3 : 6.$$

This second sequence can be interpreted as the number of independent geometric directions in point, line, area, and volume stages of a three-dimensional simplex-like embedding. It is not the branch count.

Remark 7.1 (Why the distinction matters). The branch count enters the polynomial theorem. It explains why the constant term of $N_3(\lambda)$ is 7 rather than 6. The geometric direction count may motivate the decimal scale because

$$0 + 1 + 3 + 6 = 10,$$

but it is not used to prove the polynomial formula.

The operator theory below gives a second theorem-level source of 10: the complex dimension ladder $(1, 2, 3, 4)$ of an exact rank-one operator tower. This operator ladder, not the geometric direction ladder, is used to select $\lambda_{\text{op}} = 10$.

7.2. Overall-scale gauge.

Definition 7.2 (Unnormalized multiplicative valuation). Fix $n \in \mathbb{N}$ and $\lambda > 0$. An unnormalized multiplicative valuation on P_n is a function

$$\widetilde{W}_{n,\lambda} : \text{Ob}(P_n) \rightarrow \mathbb{R}_{>0}$$

for which there exists $c > 0$ such that

$$\widetilde{W}_{n,\lambda}(L_d) = c\lambda^{n-d}, \quad 0 \leq d \leq n.$$

Definition 7.3 (Overall-scale gauge equivalence). Two unnormalized valuations with the same λ are overall-scale gauge equivalent if there exists $g > 0$ such that

$$\widetilde{W}'_{n,\lambda}(L_d) = g\widetilde{W}_{n,\lambda}(L_d)$$

for all d .

Proposition 7.4 (Unique normalized representative). *Each overall-scale gauge class contains a unique representative satisfying terminal normalization $W(L_n) = 1$; that representative is $W_{n,\lambda}$.*

Proof. If $\widetilde{W}_{n,\lambda}(L_d) = c\lambda^{n-d}$, choose $g = c^{-1}$. The resulting representative has top-layer value 1 and hence equals $W_{n,\lambda}$. $\square$

Overall-scale gauge removes a common multiplicative ambiguity. It does not select the representation-scale parameter λ .

7.3. Representation-scale freedom.

Definition 7.5 (Representation-scale action). Let

$$\mathcal{W}_3 = \{W_{3,\lambda} : \lambda > 0\}$$

be the family of normalized multiplicative valuations on P_3 . Define an action of $\mathbb{R}_{>0}$ on $\mathcal{W}_3$ by

$$r \star W_{3,\lambda} = W_{3,r\lambda}, \quad r > 0.$$

Proposition 7.6 (Free and transitive action). *The action is free and transitive. Equivalently, for any $\lambda, \mu > 0$ there is a unique $r > 0$ such that*

$$r \star W_{3,\lambda} = W_{3,\mu},$$

namely $r = \mu/\lambda$.

Proof. Immediate from the definition. $\square$

A separate operator principle is therefore required to choose a distinguished normalized valuation. That principle will be the complex trace scale of a canonically complexified exact rank-one operator tower.

8. ADMISSIBLE HISTORIES AND REAL HILBERT FIBERS

8.1. Support-level admissible histories. Let **State** be a cartesian support category and let

$$M = \Sigma^*$$

be the free monoid generated by a set Σ of elementary recurrence acts. An admissibility law selects a submonoid

$$M_{\text{adm}} \subseteq M.$$

The associated admissible-history monad is the writer monad

$$T_{\text{adm}}(X) = M_{\text{adm}} \times X.$$

Proposition 8.1 (Explicit algebra laws). *A map $a : T_{\text{adm}}A \rightarrow A$ is a T_{adm} -algebra if and only if the family*

$$a_u(x) = a(u, x), \quad u \in M_{\text{adm}},$$

satisfies

$$a_\varepsilon = \text{Id}_A, \quad a_{uv} = a_u \circ a_v$$

for all $u, v \in M_{\text{adm}}$.

Proof. This is the Eilenberg-Moore algebra identity for the writer monad. The unit law gives $a(\varepsilon, x) = x$. The multiplication law gives $a(u, a(v, x)) = a(uv, x)$. $\square$

The support-level algebra says only that admissible histories compose. A quantum or Hamiltonian interpretation requires a Hilbert-fiber realization.

8.2. Real Hilbert-fiber realization.

Assumption 8.2 (Real Hilbert-fiber realization). There is a symmetric monoidal realization functor

$$\mathcal{H}_{\mathbb{R}} : \mathbf{State}_{\text{loc}} \longrightarrow \mathbf{FHilb}_{\mathbb{R}}$$

from finite local supports to finite-dimensional real Hilbert spaces, compatible with finite disjoint unions. A finite lattice complex $L = (V, E, F)$ carries a real Hilbert fiber of the form

$$H_{L, \mathbb{R}} \cong \bigotimes_{x \in V} H_{x, \mathbb{R}}^{\text{mat}} \otimes \bigotimes_{\ell \in E} H_{\ell, \mathbb{R}}^{\text{gauge}},$$

where tensor products are real Hilbert tensor products.

The base/fiber separation is important. The support category remains cartesian and suitable for monads, completion, and projection. The real Hilbert fiber carries local dynamics, constraints, and eventual complexification.

8.3. History operators. Let $\Sigma_{\text{dyn}} \subseteq \Sigma$ be the dynamic generator alphabet relevant to a finite support L . Suppose that each $s \in \Sigma_{\text{dyn}}$ is assigned a bounded real-linear operator

$$W_s^L \in \mathcal{B}(H_{L,\mathbb{R}}).$$

For a word $u = s_1 \cdots s_m$ define

$$U_u^L = W_{s_1}^L \cdots W_{s_m}^L, \quad U_\varepsilon^L = I.$$

Proposition 8.3 (History representation from local generator data). *The assignment $u \mapsto U_u^L$ is a representation of the admissible-history monoid on $H_{L,\mathbb{R}}$:*

$$U_{uv}^L = U_u^L U_v^L, \quad U_\varepsilon^L = I.$$

If every W_s^L is orthogonal, then each U_u^L is orthogonal.

Proof. The identity statement is by definition. If $u = s_1 \cdots s_m$ and $v = t_1 \cdots t_k$, then

$$U_{uv}^L = W_{s_1}^L \cdots W_{s_m}^L W_{t_1}^L \cdots W_{t_k}^L = U_u^L U_v^L.$$

Products of orthogonal operators are orthogonal. $\square$

8.4. Gauge-admissible physical histories. Assume that $H_{L,\mathbb{R}}$ carries a closed real physical subspace

$$H_{L,\mathbb{R}}^{\text{phys}} \subseteq H_{L,\mathbb{R}}$$

with orthogonal projection P_G^L encoding finite-volume gauge constraints.

Definition 8.4 (Gauge-admissible history). A dynamic history $u \in M_{\text{adm}}$ is gauge-admissible at L if

$$U_u^L(H_{L,\mathbb{R}}^{\text{phys}}) \subseteq H_{L,\mathbb{R}}^{\text{phys}}.$$

Equivalently,

$$P_G^L U_u^L P_G^L = U_u^L P_G^L.$$

Let M_L^{phys} denote the set of such histories.

Proposition 8.5 (Physical histories form a submonoid). *M_L^{phys} is a submonoid of M_{adm} .*

Proof. The empty history acts as the identity and preserves the physical subspace. If u and v preserve it, then $U_{uv}^L = U_u^L U_v^L$ preserves it as well. $\square$

8.5. Observable descent. Observable dynamics is obtained after completion and outer projection. At fixed support L , represent this by a surjective map of sets

$$\pi_L : H_{L,\mathbb{R}}^{\text{phys}} \longrightarrow O_L,$$

where O_L is an outer observable state space. Equivalently, O_L is the quotient by the equivalence relation $\psi \sim \phi$ if and only if $\pi_L(\psi) = \pi_L(\phi)$.

Assumption 8.6 (Projection compatibility). For each completed cycle history u in a submonoid $C_L \subseteq M_L^{\text{phys}}$, the map $\pi_L \circ U_u^L$ is constant on fibers of π_L .

Theorem 8.7 (Observable semigroup descent). *Under projection compatibility, each $u \in C_L$ induces a unique observable map*

$$\bar{U}_u^L : O_L \rightarrow O_L$$

such that

$$\pi_L \circ U_u^L = \bar{U}_u^L \circ \pi_L.$$

Moreover,

$$\bar{U}_{uv}^L = \bar{U}_u^L \circ \bar{U}_v^L, \quad \bar{U}_\varepsilon^L = \text{Id}_{O_L}.$$

Proof. Since $\pi_L \circ U_u^L$ is constant on fibers of π_L , it factors uniquely through the quotient O_L . The semigroup identity follows from

$$\pi_L \circ U_{uv}^L = \pi_L \circ U_u^L U_v^L = \bar{U}_u^L \circ \pi_L \circ U_v^L = \bar{U}_u^L \circ \bar{U}_v^L \circ \pi_L,$$

and uniqueness of the factorization. $\square$

Proposition 8.8 (Generic non-invertibility under merging). *Suppose there exist $\psi, \phi \in H_{L, \mathbb{R}}^{\text{phys}}$ and $u \in C_L$ such that*

$$\pi_L(\psi) \neq \pi_L(\phi), \quad \pi_L(U_u^L \psi) = \pi_L(U_u^L \phi).$$

Then $\bar{U}_u^L$ is not injective and hence not invertible.

Proof. By descent,

$$\bar{U}_u^L(\pi_L(\psi)) = \pi_L(U_u^L \psi) = \pi_L(U_u^L \phi) = \bar{U}_u^L(\pi_L(\phi)).$$

The inputs are distinct, so $\bar{U}_u^L$ is not injective. $\square$

Interpretation 8.9. This is the mathematical slot for irreversible outer dynamics. Inner admissible histories may act reversibly on the physical Hilbert fiber while outer projection merges hidden chronology and produces a non-invertible observable semigroup.

9. CANONICAL COMPLEXIFICATION OF REAL HILBERT-FIBER DYNAMICS

The preceding section is deliberately real. We now state the canonical-complexification machinery that turns a positive real dynamics into a complex quantum sector without assuming a scalar i at the start.

9.1. The Real commutant. Let $H_{\mathbb{R}}$ be a real Hilbert space, $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification, and C the canonical conjugation. Let $U : G \rightarrow O(H_{\mathbb{R}})$ be an orthogonal representation and let V be its complexification.

Definition 9.1 (Real commutant). The Real commutant of U is

$$A_{\mathbb{R}}(U) = \{T \in \mathcal{B}(H_{\mathbb{C}}) : TV(g) = V(g)T \text{ for all } g \in G, TC = CT\}.$$

Definition 9.2 (Admissible imaginary units). The symmetry-compatible orthogonal complex structures are

$$\mathcal{X}(U) = \{J \in \mathcal{B}(H_{\mathbb{R}}) : JU(g) = U(g)J, J^* = -J, J^2 = -I\}.$$

Proposition 9.3 (Real-commutant classification). *Complexification induces an isometric real $*$ -isomorphism*

$$\{S \in \mathcal{B}(H_{\mathbb{R}}) : SU(g) = U(g)S \ \forall g \in G\} \longrightarrow A_{\mathbb{R}}(U).$$

Consequently, $\mathcal{X}(U)$ is canonically identified with the skew-adjoint square roots of $-I$ in $A_{\mathbb{R}}(U)$.

Proof. If S is a bounded real-linear intertwiner, its complexification is complex-linear, commutes with $V(g)$, and commutes with C , hence belongs to $A_{\mathbb{R}}(U)$. Conversely, if $T \in A_{\mathbb{R}}(U)$, then $TC = CT$, so T preserves $\text{Fix}(C) = H_{\mathbb{R}}$. The restriction $T|_{H_{\mathbb{R}}}$ is a bounded real-linear intertwiner. The two maps are inverse and preserve norm and adjoint. The final statement follows because an orthogonal complex structure is exactly a skew-adjoint real-linear square root of $-I$. $\square$

This proposition is the compatibility filter for all imaginary units in the operator layer. It prevents complex structure from being an arbitrary coordinate choice.

9.2. Positive-energy selection.

Theorem 9.4 (Positive-energy canonical complexification). *Let $U(t) = e^{tA}$ be a strongly continuous orthogonal group on a real Hilbert space $H_{\mathbb{R}}$, with real skew-adjoint generator A . Let the complexified unitary group be*

$$V(t) = e^{itK}$$

with $K = K^*$ and spectral measure E . Then

$$CKC = -K, \quad CE(\Delta) = E(-\Delta)C.$$

Define

$$H_{\mathbb{R}}^0 = H_{\mathbb{R}} \cap E(\{0\})H_{\mathbb{C}}, \quad H_{\mathbb{R}}^{\text{dyn}} = \{x + Cx : x \in E((0, \infty))H_{\mathbb{C}}\},$$

and

$$J = i(E((0, \infty)) - E((-\infty, 0)))|_{H_{\mathbb{R}}}, \quad B = |K| |_{H_{\mathbb{R}}}.$$

Then

$$H_{\mathbb{R}} = H_{\mathbb{R}}^0 \oplus H_{\mathbb{R}}^{\text{dyn}}$$

orthogonally, J vanishes on $H_{\mathbb{R}}^0$, and J is a bounded orthogonal complex structure on $H_{\mathbb{R}}^{\text{dyn}}$. On the natural domain in $H_{\mathbb{R}}^{\text{dyn}}$,

$$A = JB = BJ, \quad B \geq 0.$$

Proof. Since C is antiunitary and commutes with the real flow, it reverses the Stone generator on the complexification: $CKC = -K$. Hence it exchanges the positive and negative spectral subspaces and preserves the zero spectral subspace. Every real vector splits uniquely into a zero part and a vector $x + Cx$ with x positive spectral. On such a vector,

$$J(x + Cx) = i(x - Cx),$$

so $J^2 = -I$ and $J^* = -J$ on the dynamical sector. The complexification of A is iK , while the complexification of J is the spectral multiplier i on $(0, \infty)$ and $-i$ on $(-\infty, 0)$. Thus $J|K| = iK$ on the nonzero complex spectral sector. Restricting to real vectors gives $A = JB = BJ$. $\square$

Theorem 9.5 (Positive-energy rigidity). *Let J' be another bounded orthogonal complex structure on $H_{\mathbb{R}}^{\text{dyn}}$ commuting with the real flow. Suppose there exists a positive self-adjoint operator B' such that*

$$A = J'B' = B'J'$$

on the natural domain. Then

$$J' = J, \quad B' = B|_{H_{\mathbb{R}}^{\text{dyn}}}.$$

Proof. Because $(J')^* = -J'$ and J' commutes with B' ,

$$A^*A = (-B'J')(J'B') = (B')^2.$$

Since $B' \geq 0$, uniqueness of the positive square root gives $B' = (A^*A)^{1/2}$. The factorization from Theorem 9.4 gives the same positive square root B . Thus $B' = B$. Now $JBx = Ax = J'Bx$ for $x \in \text{Dom}(B) \cap H_{\mathbb{R}}^{\text{dyn}}$. The kernel of B on the nonzero-energy sector is trivial, so the range of B is dense there. Since J and J' are bounded, they agree everywhere. $\square$

Corollary 9.6 (No alternative positive-energy imaginary unit). *Within the positive-energy factorization class of a fixed real flow, the imaginary unit is unique. Any alternative presentation of the same real dynamics must change the real generator, fail symmetry compatibility, abandon positive energy, fail positivity of the chosen covariance, or lie in an exceptional zero, time-reversing, or quaternionic sector not oriented by the positive spectral split.*

9.3. Application to finite supports. Apply the preceding theorem to the real physical fiber $H_{L,\mathbb{R}}^{\text{phys}}$ whenever a calibrated completed-cycle family gives a strongly continuous orthogonal group

$$V_L(t) = e^{tA_L}.$$

If the nonzero spectral sector is nontrivial, canonical complexification selects a unique positive-energy complex structure

$$J_L : H_{L,\mathbb{R}}^{\text{dyn}} \rightarrow H_{L,\mathbb{R}}^{\text{dyn}}$$

inside the positive-energy factorization class. The resulting complex Hilbert space is denoted

$$H_{L,J_L}^{\text{dyn}}.$$

Operators preserving J_L become complex-linear operators on this one-particle sector.

Definition 9.7 (J -linear and J -defective operators). Let J be a selected complex structure on a real Hilbert space $H_{\mathbb{R}}$. A bounded real operator T is J -linear if $TJ = JT$. Its complex-linearity defect is

$$\delta_J(T) = [T, J].$$

The theorem-level scale selection below uses J -linear internal and reconstruction operators. Nonzero defects are not ignored; they are downstream correction data and may contribute to Δ .

10. CALIBRATED CYCLES AND HAMILTONIAN GENERATION

10.1. Real calibrated cycles.

Assumption 10.1 (Cycle calibration). For a finite support L , a completed-cycle family is a family $\{V_L(t)\}_{t \geq 0} \subseteq \mathcal{B}(H_{L,\mathbb{R}}^{\text{phys}})$ such that

- (i) $V_L(0) = I$;
- (ii) $V_L(t+s) = V_L(t)V_L(s)$ for $s, t \geq 0$;
- (iii) $t \mapsto V_L(t)\psi$ is norm-continuous for every ψ ;
- (iv) the family is obtained as the calibrated completed-cycle evolution of the underlying history monoid, either by a scaling limit of repeated cycles or by an exact outer-time assignment.

Theorem 10.2 (Generator theorem for calibrated cycles). *Assume $\{V_L(t)\}_{t \geq 0}$ satisfies cycle calibration.*

- (i) *If each $V_L(t)$ is contractive, then there exists a closed densely defined generator G_L such that $V_L(t) = e^{tG_L}$ in the sense of strongly continuous semigroups.*
- (ii) *If the family extends to a strongly continuous orthogonal group $t \in \mathbb{R}$, then there is a real skew-adjoint generator A_L such that $V_L(t) = e^{tA_L}$.*
- (iii) *In the group case, canonical complexification of A_L selects J_L and $B_L \geq 0$ on the nonzero-energy sector, and $A_L = J_L B_L = B_L J_L$.*

Proof. Part (1) is the Hille-Yosida generation theorem for strongly continuous contraction semigroups. Part (2) is Stone's theorem in the real orthogonal form. Part (3) is Theorem 9.4 applied to the real physical fiber. $\square$

10.2. Post-complexification local Hamiltonian cycles. Once J_L has been selected, one may use ordinary complex Hamiltonian language on H_{L,J_L}^{dyn} . Let

$$H = H_M + H_{\text{hop}} + H_E + H_B$$

be a bounded self-adjoint local Hamiltonian decomposition on a finite-dimensional J_L -complex physical sector, and suppose each block preserves the physical sector. Define the completed cycle operator

$$U_{\Delta t}^{\text{cyc}} = e^{-i\Delta t H_M} e^{-i\Delta t H_{\text{hop}}} e^{-i\Delta t H_E} e^{-i\Delta t H_B}.$$

Here multiplication by i means the selected operator J_L .

Theorem 10.3 (Finite-volume gauge-admissible cycle). *Under the preceding assumptions:*

- (i) $U_{\Delta t}^{\text{cyc}}$ is unitary on the selected complex physical sector;
- (ii) in operator norm,

$$U_{\Delta t}^{\text{cyc}} = e^{-i\Delta t H} + O(\Delta t^2), \quad \Delta t \rightarrow 0;$$

- (iii) support-level propagation bounds attached to the completed-cycle word pass to this operator realization.

Proof. Each exponential is unitary and preserves the physical sector, so the product is unitary there. The first-order Lie-Trotter product formula in the bounded setting gives the norm estimate. The propagation statement is inherited from the support-level history law by construction of the cycle from the same completed word. $\square$

11. OBSERVABLE DESCENT AND COMPLEX-STRUCTURE COMPATIBILITY

Outer projection can destroy reversibility and can also destroy complex linearity. Canonical complexification makes this test precise.

Definition 11.1 (J -compatible observable quotient). Let J_L be the selected complex structure on $H_{L,\mathbb{R}}^{\text{dyn}}$. A quotient map

$$\pi_L : H_{L,\mathbb{R}}^{\text{dyn}} \rightarrow O_L$$

is J_L -compatible if the equivalence relation $\psi \sim \phi$ defined by $\pi_L(\psi) = \pi_L(\phi)$ satisfies

$$\psi \sim \phi \implies J_L \psi \sim J_L \phi.$$

Proposition 11.2 (Descent of the selected complex structure). *If π_L is J_L -compatible, then J_L descends to a well-defined complex structure $\bar{J}_L$ on the quotient vector space whenever O_L is given the corresponding linear quotient structure. If U_u^L is J_L -linear and descends to $\bar{U}_u^L$, then $\bar{U}_u^L$ is $\bar{J}_L$ -linear.*

Proof. Define $\bar{J}_L[\psi] = [J_L \psi]$. J_L -compatibility makes this independent of the representative. If $U_u^L J_L = J_L U_u^L$, then on quotient classes

$$\bar{U}_u^L \bar{J}_L[\psi] = [U_u^L J_L \psi] = [J_L U_u^L \psi] = \bar{J}_L \bar{U}_u^L[\psi].$$

$\square$

Remark 11.3 (Projection defects). If the quotient is not J_L -compatible, then the selected complex quantum structure lives on the inner physical sector but not on the observable quotient. This is a mathematically precise defect. It is a natural candidate contributor to a downstream correction term Δ , but no scalar formula for that correction is derived here.

12. MINIMAL EXACT PROJECTION TOWERS IN THE SELECTED COMPLEX SECTOR

The complex structure used in this section is the positive-energy structure selected by Theorem 2.2. Thus the dimensions below are complex dimensions in a canonically selected sector, rather than dimensions relative to an arbitrary choice of scalar i .

12.1. Exact rank-one towers. Let H_{L,J_L}^{dyn} be the selected complex Hilbert space of a finite support L . For a J_L -invariant real subspace $K \subseteq H_{L,\mathbb{R}}^{\text{dyn}}$, write $\dim_{J_L} K$ for its dimension as a complex vector space under J_L .

Definition 12.1 (Canonically complexified exact rank-one operator tower). A canonically complexified exact rank-one operator tower is a finite-dimensional J_L -invariant graded subspace

$$K = \bigoplus_{d=0}^3 K_d \subseteq H_{L,\mathbb{R}}^{\text{dyn}}$$

together with bounded J_L -linear operators

$$T_i, T_o \in \mathcal{B}(K)$$

satisfying:

- (RG1) $T_i(K_d) \subseteq K_d$ for $d = 0, 1, 2, 3$;
- (RG2) $T_o(K_d) \subseteq K_{d+1}$ for $d = 0, 1, 2$, and $T_o(K_3) = 0$;
- (RG3) $T_o|_{K_d} : K_d \rightarrow K_{d+1}$ is injective for $d = 0, 1, 2$;
- (RG4) with $\Omega = [T_i, T_o]$, one has $\Omega(K_d) \subseteq K_{d+1}$ for $d = 0, 1, 2$;
- (RG5) for $d = 0, 1, 2$ the quotient sequence

$$0 \longrightarrow K_d \xrightarrow{T_o} K_{d+1} \longrightarrow Q_d \longrightarrow 0$$

is exact in finite-dimensional complex vector spaces over the selected complex structure, and $Q_d \cong \mathbb{C}$;

- (RG6) $\dim_{J_L} K_0 = 1$.

Remark 12.2 (Why complex dimensions are used). If K_d is invariant under a complex structure, its real dimension is even unless it is zero. The ladder below is therefore a complex dimension ladder after canonical complexification.

Proposition 12.3 (Cyclic realization of the exact tower). *Let C be a J -linear operator on a complex vector space and let ω be a vector such that*

$$\omega, C\omega, C^2\omega, C^3\omega$$

are linearly independent. Define

$$F_d = \text{Span}_{\mathbb{C}}\{\omega, C\omega, \dots, C^d\omega\}, \quad d = 0, 1, 2, 3.$$

Then the inclusions $F_d \hookrightarrow F_{d+1}$ form exact sequences

$$0 \longrightarrow F_d \longrightarrow F_{d+1} \longrightarrow \mathbb{C} \longrightarrow 0, \quad d = 0, 1, 2.$$

Consequently a single cyclic length-four operator supplies a model for the rank-one exactness condition.

Proof. The displayed vectors form a basis of F_d for $0 \leq d \leq 3$. Hence $\dim F_{d+1} = \dim F_d + 1$ and F_{d+1}/F_d is one-dimensional. $\square$

12.2. The dimension ladder.

Theorem 12.4 (Dimension ladder from an exact rank-one tower). *Let $K = \bigoplus_{d=0}^3 K_d$ be a canonically complexified exact rank-one operator tower. Then*

$$\dim_{J_L} K_d = d + 1, \quad d = 0, 1, 2, 3.$$

Consequently,

$$\dim_{J_L} K = 1 + 2 + 3 + 4 = 10.$$

Proof. Exactness gives

$$\dim_{J_L} K_{d+1} = \dim_{J_L} T_o(K_d) + \dim_{J_L} Q_d.$$

The map $T_o|_{K_d}$ is injective and J_L -linear, so $\dim_{J_L} T_o(K_d) = \dim_{J_L} K_d$. Since $Q_d \cong \mathbb{C}$, $\dim_{J_L} Q_d = 1$. Hence

$$\dim_{J_L} K_{d+1} = \dim_{J_L} K_d + 1.$$

Starting from $\dim_{J_L} K_0 = 1$ gives 1, 2, 3, 4, and summing the four layer dimensions gives 10. $\square$

Corollary 12.5 (Simplicial realization). *For each $d \in \{0, 1, 2, 3\}$ there exists an abstract d -simplex Δ^d with vertex set V_d of cardinality $d+1$ and a complex vector-space isomorphism*

$$K_d \cong \mathbb{C}[V_d]$$

relative to the selected complex structure J_L .

Proof. Both sides have complex dimension $d + 1$. $\square$

12.3. Operator normalization scale.

Definition 12.6 (Complex trace scale). Let K be a canonically complexified exact rank-one operator tower. Its operator normalization scale is

$$\lambda_{\text{op}} := \text{Tr}_{J_L}(I_K) = \dim_{J_L} K,$$

where Tr_{J_L} is the trace computed over the complex Hilbert space defined by J_L .

Definition 12.7 (Canonically complexified operator-admissible valuation). A normalized valuation $W_{3,\lambda}$ is canonically complexified operator-admissible if

$$\lambda = \lambda_{\text{op}}$$

for some canonically complexified exact rank-one operator tower.

Theorem 12.8 (Canonical operator-selected scale). *If the real admissible-history Hilbert-fiber system contains a canonically complexified exact rank-one operator tower, then*

$$\lambda_{\text{op}} = 10.$$

Hence the unique canonically complexified operator-admissible normalized valuation is

$$W_{3,10}.$$

Proof. By Theorem 12.4, every such sector has complex dimension 10. Therefore $\lambda_{\text{op}} = 10$ by definition. Operator-admissibility then forces $\lambda = 10$. $\square$

Corollary 12.9 (Compatibility with valuation scale action). *Every normalized valuation $W_{3,\lambda}$ is carried by a unique representation-scale renormalization to the operator-admissible valuation $W_{3,10}$, namely*

$$r = 10/\lambda.$$

Proof. This is the free and transitive scale action of Proposition 7.6. $\square$

13. STRUCTURAL COROLLARY: THE INTEGER 137

Theorem 5.2 is the compact post-complexification statement. We now spell out the final consequence in the language of reconstruction polynomials.

Main Theorem 13.1 (The structural integer 137). *Assume:*

- (i) *a real Hilbert representation or real orthogonal flow whose nonzero-energy sector carries the positive-energy complex structure J of Theorem 2.2;*
- (ii) *a height-three projection category P_3 with branch-count functor B_3 and normalized valuation $W_{3,\lambda}$;*
- (iii) *an exact J -linear one-innovation projection tower $K = \bigoplus_{d=0}^3 K_d$ with $\dim_J K_0 = 1$;*
- (iv) *the operator-admissibility condition $\lambda = \text{Tr}_J(I_K)$.*

Then the scale is uniquely

$$\lambda_{\text{op}} = 10,$$

the operator-admissible valuation is $W_{3,10}$, and the corresponding bare reconstruction value is

$$N_3(\lambda_{\text{op}}) = N_3(10) = 137.$$

Moreover this value is invariant under replacement of the selected complex structure by any other positive-energy factorization of the same real flow, because positive-energy rigidity forces the replacement to coincide with J .

Proof. The exact tower gives $\mathrm{Tr}_J(I_K) = 10$ by Theorem 12.8. The reconstruction polynomial gives $N_3(\lambda) = \lambda^2 + 3\lambda + 7$, so

$$N_3(10) = 100 + 30 + 7 = 137.$$

Rigidity of J follows from Theorem 9.5, equivalently from the imported positive-energy uniqueness theorem. $\square$

Remark 13.2 (Exact scope). The theorem proves uniqueness of a bare structural value inside a stated post-complexification projection class. It does not prove that the measured electromagnetic coupling is $1/137$, and it does not compute radiative, transport, or renormalization corrections. Those effects belong to Δ in Section 14.

Layer d	Geom. count	Branch count	$\dim_J K_d$	Contrib. at $\lambda = 10$
0	0	0	1	0
1	1	1	2	100
2	3	3	3	30
3	6	7	4	7
Total	10	11	10	137

The table shows the separation of roles. The geometric ladder can motivate a decadic embedding. The exact operator ladder selects the scale 10. The branch ladder supplies the coefficients of the polynomial.

13.1. Exclusion of nearby alternatives. The theorem excludes nearby alternatives only inside the post-complexification projection class.

- Changing the height changes the polynomial family. For example,

$$N_2(10) = 13, \quad N_4(10) = 10^3 + 3 \cdot 10^2 + 7 \cdot 10 + 15 = 1385.$$

- Changing the operator tower changes the trace scale. A tower with different cokernel dimensions would give a different $\mathrm{Tr}_J(I_K)$.
- Changing the branch functor changes the coefficients of N_3 .
- Changing the valuation scale away from 10 is allowed algebraically, but it is not operator-admissible for the selected exact tower.
- Changing the positive-energy complex structure is not available within the same real flow and positive-energy factorization class, because canonical complexification is rigid.

13.2. Relation to earlier weighted projection-sum diagrams. Earlier projection diagrams sometimes displayed a weighted sum of the form

$$10^2 + 3 \cdot 10 + 7 = 137.$$

The present formulation explains its theorem-level status. The coefficients 1, 3, 7 are branch-count coefficients; the scale 10 is selected by the exact J -linear operator tower; and the equality is the specialization of the polynomial $N_3(\lambda)$ at $\lambda = 10$. Thus the primitive object is the polynomial and the selected trace scale, not a decimal trick.

14. PROJECTED COUPLINGS AND CORRECTION SLOTS

The structural value 137 is a bare reconstruction value. A physical coupling requires a correction or matching layer.

Definition 14.1 (Projected model coupling). Let $\Delta \in \mathbb{R}$ be a scalar correction term. Define

$$A_{\text{proj}}(\lambda, \Delta) = N_3(\lambda) + \Delta, \quad \alpha_{\text{proj}}(\lambda, \Delta) = \frac{1}{N_3(\lambda) + \Delta}.$$

Proposition 14.2 (Distinguished specialization). *At the operator-selected scale,*

$$\alpha_{\text{proj}}(10, \Delta) = \frac{1}{137 + \Delta}.$$

Proof. Use $N_3(10) = 137$. □

Corollary 14.3 (Small-correction expansion). *If $|\Delta| \ll N_3(\lambda)$, then*

$$\alpha_{\text{proj}}(\lambda, \Delta) = \frac{1}{N_3(\lambda)} - \frac{\Delta}{N_3(\lambda)^2} + O(\Delta^2).$$

In particular,

$$\alpha_{\text{proj}}(10, \Delta) = \frac{1}{137} - \frac{\Delta}{137^2} + O(\Delta^2).$$

Remark 14.4 (Permitted sources of Δ). The present paper does not derive Δ . It identifies possible mathematical slots:

- projection defect, if outer projection is many-to-one or not J_L -compatible;
- transport or holonomy defect in a fibred gauge model;
- amplitude-level corrections from coherent sums over admissible histories;
- renormalization or regulator removal effects in a finite-volume gauge model.

None of these corrections is folded back into the polynomial theorem.

15. FROM THE STRUCTURAL VALUE TO A PHYSICAL COUPLING

The preceding section introduces a scalar correction slot, but a scalar slot is not a physical derivation. The measured inverse fine-structure constant is not the integer 137; the CODATA 2022 recommended value is

$$\alpha^{-1} = 137.035999177(21),$$

so any connection between the present structural integer and electromagnetism must account for the nonzero offset

$$\Delta_{\text{obs}} \approx 0.035999177.$$

In the notation of Section 14, a physical matching theorem would have to explain why the electromagnetic sector produces

$$\alpha_{\text{phys}}^{-1} = 137 + \Delta_{\text{phys}}$$

with Δ_{phys} equal, within controlled uncertainties and at a specified energy/renormalization convention, to the observed correction.

The present paper supplies only the bare structural term. To obtain a statement about the electromagnetic fine-structure constant one would need, at a minimum, a finite-regulator gauge model with a distinguished electromagnetic observable sector, a controlled continuum or infrared limit, and a proof that the projected coupling extracted from that sector has the form above with a computable correction term.

A possible matching theorem would have to establish the following chain.

- (M1) Construct a finite-regulator projected gauge model whose real Hilbert dynamics satisfies the hypotheses of the canonical-complexification theorem and whose selected positive-energy sector contains the exact one-innovation projection tower used here.
- (M2) Prove reflection positivity, transfer-matrix positivity, or an equivalent reconstruction theorem, yielding a physical Hilbert space, vacuum sector, and nonnegative Hamiltonian.
- (M3) Identify a gauge-invariant electromagnetic sector and a renormalized charge observable in that sector.
- (M4) Show that transport, holonomy, projection defects, J -compatibility defects, and regulator removal combine into a scalar correction Δ_{phys} in the projected inverse coupling.
- (M5) Prove that the running of the resulting coupling agrees with the appropriate low-energy effective quantum field theory in its domain of validity.

None of these items is assumed in the proof of Theorem 5.2. They are the conditions under which the structural integer 137 could become the bare term in a physical matching theorem.

Theorem Target 15.1 (Physical matching theorem). Construct a finite-regulator projected gauge model satisfying (M1)–(M5), compute Δ_{phys} , and prove that the measured electromagnetic coupling in the relevant regime is obtained from the projected inverse coupling $137 + \Delta_{\text{phys}}$ up to controlled regulator, renormalization, and infrared errors.

Remark 15.2 (Why the correction must not be hidden in the theorem). If Δ were inserted into the reconstruction polynomial itself, the structural theorem would lose its content. The polynomial theorem fixes $N_3(10) = 137$. The physical problem is a separate matching problem that determines how a regulator-dependent observable sector deforms the bare inverse coupling to $137 + \Delta_{\text{phys}}$.

Remark 15.3 (Numerical size of the physical offset). Using the CODATA 2022 recommended inverse fine-structure constant [3],

(5)

$$\beta_{\text{obs}} := \alpha_{\text{obs}}^{-1} = 137.035999177(21), \quad \Delta_{\text{obs}} := \beta_{\text{obs}} - 137 = 0.035999177(21).$$

The present structural theorem fixes the integer part of the inverse coupling in the selected sector. The observed offset is small on the inverse-coupling scale but much larger than current metrological uncertainty; it therefore

cannot be ignored or hidden in notation. It must be produced by a separate matching theorem.

16. RE-CREATION KINEMATICS, HYDROGEN, AND CONJUGATE SECTORS

This section is an interpretation and model-building layer. It is not used in the proof of Theorem 5.2. Its purpose is to record a disciplined interface between the structural integer, the simplest electromagnetic bound state, and a possible conjugate-sector extension. We separate four levels:

- (L1) the theorem-level structural core: canonical complexification, exact projection tower, and $N_3(10) = 137$;
- (L2) the standard hydrogenic fact that the fine-structure constant is the Coulomb speed ratio v/c in the Bohr–Sommerfeld ground state;
- (L3) the model ansatz that a massive projected excitation can be described by an effective re-creation lag N_{eff} ;
- (L4) the optional conjugate-sector interpretation, where antiunitary orientation reversal is compared with antihydrogen or mirror-sector language.

Only (L1) is part of the theorem. Levels (L2)–(L4) describe how the theorem might be connected to physical hydrogen and to a mirror or antiworld picture in a future regulator model.

16.1. Hydrogen and the standard speed identity. Let m_e be the electron mass, m_p the proton mass, and

$$\mu = \frac{m_e m_p}{m_e + m_p}$$

the reduced mass of the electron–proton two-body problem. In the nonrelativistic Coulomb approximation the relative coordinate r has potential

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}.$$

Proposition 16.1 (Bohr–Sommerfeld speed identity). *For the circular $n = 1$ Bohr–Sommerfeld orbit of a hydrogenic electron–proton system, the relative speed satisfies*

$$\frac{v_{\text{rel}}}{c} = \alpha_{\text{phys}} := \frac{e^2}{4\pi\epsilon_0 \hbar c}.$$

For finite proton mass, the electron speed in the center-of-mass frame is

$$v_e = \frac{\mu}{m_e} v_{\text{rel}} = \frac{m_p}{m_p + m_e} \alpha_{\text{phys}} c.$$

In the infinite-proton-mass approximation this becomes $v_e = \alpha_{\text{phys}} c$.

Proof. The circular Coulomb equation and angular quantization are

$$\frac{\mu v_{\text{rel}}^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2}, \quad \mu v_{\text{rel}} r = \hbar.$$

Eliminating r gives

$$v_{\text{rel}} = \frac{e^2}{4\pi\epsilon_0\hbar} = \alpha_{\text{phys}}c.$$

The center-of-mass relation $v_e = (\mu/m_e)v_{\text{rel}}$ gives the finite-mass correction. $\square$

Remark 16.2 (What is standard and what is not). Proposition 16.1 is standard semiclassical hydrogen physics [37, 38]. It does not depend on the projection polynomial, on re-creation language, or on mirror-sector assumptions. It gives a safe physical anchor: in the simplest Coulomb sector, α is a speed ratio.

16.2. Photon benchmark and re-creation lag.

Assumption 16.3 (Re-creation clock ansatz). Fix a fundamental projection length $\ell_0 > 0$ and a fundamental re-creation tick $\tau_0 > 0$. The photon benchmark is the one-hop-per-tick law

$$v_\gamma = c := \frac{\ell_0}{\tau_0}.$$

A massive projected excitation has effective lag $N_{\text{eff}} > 0$ if one projected hop of length ℓ_0 closes only after N_{eff} ticks. Its effective projection speed is

$$v(N_{\text{eff}}) = \frac{\ell_0}{N_{\text{eff}}\tau_0} = \frac{c}{N_{\text{eff}}}.$$

Proposition 16.4 (Lag ratio). *Under Assumption 16.3, if a projected electromagnetic sector has effective inverse coupling*

$$N_{\text{eff}} = N_3(\lambda_{\text{op}}) + \Delta = 137 + \Delta,$$

then

$$\frac{v(N_{\text{eff}})}{v_\gamma} = \frac{1}{137 + \Delta} = \alpha_{\text{proj}}(10, \Delta).$$

In the bare structural limit $\Delta = 0$,

$$\frac{v}{c} = \frac{1}{137}.$$

Proof. The photon benchmark gives $v_\gamma = c = \ell_0/\tau_0$. A lag- N_{eff} excitation has speed $\ell_0/(N_{\text{eff}}\tau_0)$, so the ratio is $1/N_{\text{eff}}$. Substituting $N_{\text{eff}} = 137 + \Delta$ gives the stated expression. $\square$

Interpretation 16.5 (Hydrogenic matching). Combining Proposition 16.1 with Proposition 16.4 suggests the following matching problem:

$$\alpha_{\text{phys}} = \frac{v_{\text{rel}}}{c} = \frac{1}{N_{\text{eff}}}, \quad N_{\text{eff}} = 137 + \Delta_{\text{phys}}.$$

This is not a derivation of Δ_{phys} . It identifies the physical meaning that such a correction would have if a finite-regulator model matched the structural sector to hydrogen: Δ_{phys} is the correction to the bare electron/projection lag relative to the photon benchmark.

Remark 16.6 (Speed ratio versus variable speed of light). The ratio $v/c = 1/N_{\text{eff}}$ should not be read as a claim that the measured vacuum speed of light varies with the electron state. The photon benchmark c remains the limiting propagation speed in the matched observable sector. The lag model assigns a slower effective projected speed to a massive bound-state degree of freedom.

16.3. Hydrogen as a minimal neutral Coulomb sector. Hydrogen is the simplest neutral electromagnetic bound state and therefore a natural low-energy test object for any projected coupling. In the selected complex structure, a schematic finite-regulator hydrogen sector has the form

$$\mathcal{H}_H^{(+J)} = \mathcal{H}_p^{\text{int}} \otimes L^2(\mathbb{R}^3; \mathbb{C}_e^2),$$

where $L^2(\mathbb{R}^3; \mathbb{C}_e^2)$ carries the relative electron coordinate and spin, and $\mathcal{H}_p^{\text{int}}$ contains a color-singlet proton core. At the valence level,

$$p^+ = uud, \quad \frac{2}{3} + \frac{2}{3} - \frac{1}{3} = +1,$$

so the total hydrogen charge with one electron is zero.

Interpretation 16.7 (Three internal constituents and one external lepton). The proton contributes a threefold internal valence pattern, while the electron contributes the minimal external Coulomb orbital degree of freedom. This gives a physical $3 + 1$ pattern inside the simplest neutral atom. The observation is a compatibility statement only: it is not used in the proof of $N_3(10) = 137$, and it does not replace QCD or the standard electroweak description of hydrogen.

Theorem Target 16.8 (Hydrogen matching theorem). Construct a finite-regulator hydrogen sector in the selected J -complex Hilbert space, prove positivity and convergence to the Coulomb/Dirac hydrogen spectrum in the appropriate low-energy limit, and show that the effective lag extracted from the relative speed satisfies

$$N_{\text{eff}} = 137 + \Delta_{\text{phys}}$$

with Δ_{phys} equal to the regulated projection, radiative, reduced-mass, and observable-sector correction of the electromagnetic model.

16.4. Conjugate orientation sectors and antihydrogen. The mirror or antiworld language should be separated from the theorem. The theorem-level operation available in the real-Hilbert framework is antiunitary orientation reversal of the selected complex structure.

Definition 16.9 (Conjugate orientation sector). Let $K^{(+)}$ be a J_+ -linear finite projection sector. A conjugate orientation sector is a finite projection sector $K^{(-)}$ together with an antiunitary map

$$\Theta : K^{(+)} \longrightarrow K^{(-)}$$

such that

$$\Theta J_+ \Theta^{-1} = -J_-.$$

If the physical sector also carries a charge-conjugation operation on the gauge and matter labels, one may compare a hydrogen sector p^+e^- with an antihydrogen sector

$$\bar{H} = (\bar{p}^- e^+), \quad \bar{p}^- = \bar{u}\bar{u}\bar{d}.$$

Proposition 16.10 (Conjugate-sector invariance of the bare scale). *Assume that Θ maps an exact one-innovation tower $K^{(+)} = \bigoplus_{d=0}^3 K_d^{(+)}$ onto a conjugate tower $K^{(-)} = \bigoplus_{d=0}^3 K_d^{(-)}$ and preserves the branch reconstruction data. Then*

$$\dim_{J_+} K_d^{(+)} = \dim_{J_-} K_d^{(-)}, \quad \text{Tr}_{J_+}(I_{K^{(+)}}) = \text{Tr}_{J_-}(I_{K^{(-)}}) = 10,$$

and both sectors have the same bare structural value

$$N_3(10) = 137.$$

Proof. Antiunitary equivalence preserves finite dimensions after reversing the complex orientation. Exactness of the tower is transported by Θ , so the same dimension ladder 1, 2, 3, 4 holds in the conjugate sector. The branch-count polynomial is unchanged by the assumed preservation of the reconstruction data. Therefore both sectors have the same trace scale and the same value $N_3(10)$. $\square$

Remark 16.11 (What the conjugate sector does and does not prove). The operation $J \mapsto -J$ defines a mathematically precise conjugate orientation sector. It does not by itself prove the existence of a physical mirror world, nor does it identify the conjugate sector with antimatter unless the gauge representation, charge conjugation, and observable sector are supplied. In a future model, hydrogen and antihydrogen may be interpreted as opposite orientations of a shared Real-commutant phase structure. Matter–antimatter asymmetry, if present, must enter through Δ , boundary conditions, charge-sector selection, or regulator dynamics, not through the bare integer 137.

16.5. Electron mass and charge as downstream quantities. The speed-ratio interpretation fixes only a dimensionless lag. To obtain charge or mass one must add dimensional data. Once a physical coupling is matched, the elementary charge satisfies

$$e = \sqrt{4\pi\epsilon_0\hbar c \alpha_{\text{phys}}} = \sqrt{\frac{4\pi\epsilon_0\hbar c}{137 + \Delta_{\text{phys}}}}.$$

This is an identity after α_{phys} has been matched; it is not an independent derivation of e .

Likewise, if each inner tick carried an independently derived energy ϵ_0 , a lag model would give

$$m_e c^2 = N_{\text{eff}} \epsilon_0 = (137 + \Delta_{\text{phys}}) \epsilon_0.$$

Without an independent derivation of ϵ_0 , this is a parametrization of the electron rest energy rather than a prediction. This distinction is essential: the structural theorem fixes a dimensionless integer and a possible dimensionless speed ratio; dimensional constants require a regulator, a unit-setting mechanism, and a physical matching theorem.

17. LOCAL OPERATOR NETS AND AMPLITUDES

This section records export interfaces to more conventional finite-regulator quantum models. It is not used in the proof of Theorem 5.2.

Definition 17.1 (Local operator net). For a bounded region $O \subseteq L$, define

$$\mathcal{A}_L(O) = C^*(U_s^L, (U_s^L)^* : s \in \Sigma_{\text{dyn}}, \text{supp}(s) \subseteq O),$$

inside $\mathcal{B}(H_{L,J_L}^{\text{dyn}})$ when the generators are J_L -linear.

Proposition 17.2 (Isotony and locality). *The assignment $O \mapsto \mathcal{A}_L(O)$ satisfies:*

- (i) *isotony: if $O_1 \subseteq O_2$, then $\mathcal{A}_L(O_1) \subseteq \mathcal{A}_L(O_2)$;*
- (ii) *locality: if generators with disjoint supports commute, then*

$$[\mathcal{A}_L(O_1), \mathcal{A}_L(O_2)] = 0$$

whenever $O_1 \cap O_2 = \emptyset$;

- (iii) *graded locality: if the generators satisfy graded commutation relations, the corresponding graded locality statement holds.*

Proof. Isotony follows from inclusion of generating sets. Locality and graded locality hold on generators by hypothesis and are preserved under forming the generated C^* -algebras. $\square$

Definition 17.3 (Coherent history amplitudes). Let $\psi_i, \psi_f \in H_{L,J_L}^{\text{dyn}}$ and let $u \in M_L^{\text{phys}}$ be a J_L -linear gauge-admissible history. Define

$$\text{Amp}(u; \psi_i, \psi_f) = \langle \psi_f, U_u^L \psi_i \rangle_{J_L}.$$

For a history class $\mathcal{C}_{\text{adm}}(\psi_i \rightarrow \psi_f)$ on which the following series converges absolutely, define

$$K(\psi_f, \psi_i) = \sum_{u \in \mathcal{C}_{\text{adm}}(\psi_i \rightarrow \psi_f)} \text{Amp}(u; \psi_i, \psi_f).$$

These amplitudes are not used to derive the reconstruction polynomial. They specify where later transport-defect and matching mechanisms would have to be expressed.

18. FINITE-REGULATOR THEOREM TARGETS

The structural theorem becomes a candidate input for a physical gauge model only after additional finite-regulator analysis. The following are theorem targets, not results of the present paper.

Theorem Target 18.1 (Finite-regulator Gibbs measure). Fix lattice spacing $a > 0$, spatial volume parameter L , even outer Euclidean time extent T , and auxiliary compact inner-time period N_τ . Construct a compact gauge configuration space, a gauge-invariant dual-time lattice action, and a normalized gauge-invariant Gibbs measure.

Theorem Target 18.2 (Reflection positivity and OS reconstruction). Define the outer-time reflection map, the positive-time observable algebra, and the Osterwalder-Schrader sesquilinear form. Prove reflection positivity of the finite-regulator Gibbs measure and construct the corresponding Hilbert space by quotient and completion.

Theorem Target 18.3 (Transfer matrix and Hamiltonian reconstruction). Construct a bounded positive transfer operator on the reconstructed Hilbert space, together with a distinguished vacuum vector and a nonnegative self-adjoint Hamiltonian preserving the gauge-invariant reconstructed sector.

Theorem Target 18.4 (Gauge-invariant physical sector from local cycles). Choose a finite-volume dynamic generator alphabet and bounded self-adjoint local block Hamiltonians whose exponentials define a completed cycle operator of the form used above. Prove that each block preserves the gauge-invariant physical subspace and that the completed-cycle family is unitary there and admits the stated Lie-Trotter approximation to the finite-volume lattice Hamiltonian.

Theorem Target 18.5 (Projected four-dimensional observable sector). Define a gauge-invariant projected observable algebra and the corresponding projected Hilbert sector, and prove that this sector is the correct four-dimensional carrier for later low-energy matching.

Theorem Target 18.6 (Derivation of the exact rank-one tower). Derive the quotient condition

$$\dim_{J_L}(K_{d+1}/T_o K_d) = 1$$

from observable projection, record formation, locality, or a finite-regulator transfer-matrix structure, rather than postulating it as a sector condition.

19. DISCUSSION AND LIMITATIONS

The paper now has five logically separated components.

First, the dimensionality layer is conditional but precise: stable finite-dimensional projection composition, if associative, norm-preserving, and nonabelian, selects the quaternionic envelope and hence a three-dimensional vector part.

Second, the decimal-readout layer is exact once the height-three closure hierarchy with multiplicity ten is accepted: a hierarchy with closure multiplicity 10 requires a ten-symbol primitive alphabet for locally faithful elementary readout, and ten symbols are sufficient.

Third, the polynomial layer is exact and self-contained: the projection category, branch-count functor, valuation, and reconstruction polynomial prove $N_3(\lambda) = \lambda^2 + 3\lambda + 7$.

Fourth, the operator layer is concrete but conditional: admissible histories act on real Hilbert fibers, gauge-admissible histories preserve physical subspaces, calibrated cycles generate real dynamics, and positive-energy dynamics selects a unique complex structure within the positive-energy factorization class.

Fifth, the scale-selection layer is conditional on the existence of a canonically complexified exact rank-one operator tower. Once that sector exists, its complex dimension is forced to be 10, so the operator-selected bare value is forced to be 137.

The limitations are explicit.

- L1. The dimensionality theorem is conditional on stable-composition assumptions; it is not yet derived from a unique microscopic finite-regulator update law.
- L2. The decimal minimality theorem concerns primitive one-slot readout for the decadic closure hierarchy, not the historical origin or universal optimality of human decimal notation.
- L3. The polynomial theorem does not derive the measured fine-structure constant.
- L4. The operator layer does not yet derive the exact rank-one tower from a finite-regulator gauge model.
- L5. The correction term Δ is not computed, and no physical matching theorem to the measured electromagnetic coupling is proved.
- L6. No QED beta function, vacuum-polarization series, or running coupling is derived.
- L7. Reflection positivity, transfer-matrix reconstruction, and physical matching remain open theorem targets.

These limitations do not weaken the theorem; they identify its exact scope. The result is best read as a structural uniqueness theorem for a bare reconstruction value inside a positive-energy, canonically complexified, rank-one admissible operator sector.

APPENDIX A. LOW-DIMENSIONAL POLYNOMIALS

The first few reconstruction polynomials are

$$N_0(\lambda) = 0,$$

$$N_1(\lambda) = 1,$$

$$N_2(\lambda) = \lambda + 3,$$

$$N_3(\lambda) = \lambda^2 + 3\lambda + 7,$$

$$N_4(\lambda) = \lambda^3 + 3\lambda^2 + 7\lambda + 15.$$

Thus the three-dimensional case is part of a systematic family rather than an isolated arithmetic construction.

APPENDIX B. OPERATOR STATUS MAP

Layer	Status in this paper
Quaternionic dimensionality filter	Proved from stated stability assumptions plus standard classification
Topological H -space universality route	Recorded under stronger symmetry/homotopy assumptions
Minimal primitive readout theorem	Proved
Decadic scale for any hierarchy with $M(P) = 10$	Proved from primitive readout
Sexagesimal $10 \cdot 6 \cdot 6 = 360$ interpretation	Interpretive compatibility
Projection category, branch counting, valuation, polynomial	Proved
Support-level admissible-history monad	Proved
Real Hilbert-fiber realization and history representation	Proved from stated assumptions
Gauge-admissible physical submonoid	Proved
Observable semigroup descent	Proved under projection compatibility
Canonical complexification and positive-energy rigidity	Proved from standard spectral assumptions
Calibrated-cycle generator theorem	Proved from standard semigroup/group assumptions
Finite-volume completed-cycle example	Proved under bounded self-adjoint block assumptions
Exact J -linear rank-one tower dimension ladder	Proved by additivity of exact sequences
Existence of the exact rank-one operator tower	Theorem target
Reflection positivity, transfer matrix, projected sector	Theorem targets
Physical matching to measured fine-structure constant	Open
Holographic bound, interaction hierarchy, and internal gauge layers	Programmatic theorem targets

APPENDIX C. PROGRAMMATIC EXTENSIONS: HOLOGRAPHY, PROJECTION ALGEBRA, AND INTERNAL LAYERS

The following material is included to merge the broader companion projection manuscripts into the present paper while preserving claim-status discipline. None of these extensions is used in the proof of Theorem 5.2.

They indicate where the dimensionality, decimal-scale, and 137 mechanisms may interface with larger physical programs.

C.1. Holographic projection bound. A recurring idea in the projection manuscripts is that a three-dimensional region can be reconstructed from families of two-dimensional projections. In a discrete setting, the number of independent projection records required to specify a bounded three-dimensional bulk should scale like the number of boundary or surface records rather than the number of bulk cells. This supplies a combinatorial route to an area-law or holographic bound.

A conservative theorem-target formulation is the following.

Theorem Target C.1 (Projection-theoretic holographic bound). Given a finite three-dimensional projection complex with a specified family of two-dimensional boundary readouts, prove an injective or stably reconstructive map from admissible bulk configurations modulo gauge/reconstruction equivalence into boundary projection data. Show that the number of required independent records scales with boundary area, up to controlled correction terms.

This is compatible with the present paper because the same selected three-dimensional structure that supports quaternionic spatial stability also contains the two-dimensional face count $P_2 = 6$ used in surface readout and circular subdivision. However, proving a Bekenstein–Hawking law requires a gravitational entropy interpretation not supplied by the bare reconstruction polynomial.

C.2. Projection algebra for interaction hierarchy. Another companion formulation uses the symbolic projection expression

$$(nD + T)(nD - T) \longrightarrow nD$$

to organize dimensional layers and interaction types. A cautious status table is:

Layer	Projection expression	Interpretation	Candidate interaction
0	$(0D + 0T)(0D - 0T)$	Monopole or pre-physical seed	Origin
0'	$(0D + T)(0D - T)$	Time-polarized origin	Weak sector
1	$(1D + T)(1D - T)$	Line coherence	Strong sector
2	$(2D + T)(2D - T)$	Surface coherence	Electromagnetic sector
3	$(3D + T)(3D - T)$	Full spatial coherence	Gravity/spacetime

In the present paper this table is interpretive. It does not enter the proof of the decimal scale or of 137. Its possible value is as a dictionary for later finite-regulator models in which different projection layers carry different observable sectors.

C.3. Higher tick layers and internal gauge factors. The quaternionic result uses the first three stable tick projections to construct the spatial vector sector. Companion projection notes suggest that deeper layers may support internal symmetries. A schematic assignment is:

Tick layer(s)	Role	Candidate symmetry
$t_r^{(0)}-t_r^{(2)}$	Spatial quaternionic basis	SO(3) with spin cover SU(2)
$t_r^{(3)}$	Phase mode	$U(1)$
$t_r^{(4)}, t_r^{(5)}$	Doublet mode	SU(2)
$t_r^{(6)}-t_r^{(8)}$	Triplet mode	SU(3)

This assignment is not a derivation of the Standard Model in the present paper. It is an interface for future work: one would need to construct the Hilbert fibers, gauge constraints, anomaly cancellation, dynamics, and representation content from an explicit finite-regulator update model.

C.4. How these extensions should be used. The extensions above should be read as theorem targets and interpretive dictionaries. The established core of the present manuscript remains the three-filter chain:

$$3D \text{ stability} \implies 10 \text{ scale} \implies 137 \text{ branch reconstruction.}$$

Holography, interaction hierarchy, gauge multiplets, and particle spectra are natural places where the same projection machinery might continue, but they require additional analytic and physical input.

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